Theoretical Foundations of the World Input Output Tables

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The World Input-Output Databases (WIOD) are widely used to study trends in intercountry interindustry trade, mapping of global value chains, trade in value added, trade in employment, and the energy and environment sectors located in the different countries. The WIOD have also been extensively used for the analysis of intercountry intersectoral impacts of trade policies like trade liberalization and free trade agreements. In contrast to the proliferation of the policy applications of WIOD's there have been surprisingly few contributions to the theory that underlies the WIOD. Clearly trade theories of the Heckscher-Ohlin type, new trade theory or modern versions of Ricardian trade theory are insufficient to provide adequate foundations due to their limitations which include a) limited dimensionality e.g. two-countries and two-or-more goods and b) absence of intermediate capital goods. The purpose of this paper is to formulate a general equilibrium model of trade whose solution is the world input output matrix. It is shown how the static open Leontief system augmented by a household demand system (e.g. Mongelli, Neuwahl and Rueda-Cantuche, 2010) can itself be used to formulate a theory of trade between several countries in several commodities which can determine the sizes of intercountry interindustry trade flows appearing in the world input-output table in both physical and value terms.

Keywords :

1. Introduction

The World Input Output Tables (WIOT) of which the WIOD (Timmer et.al., 2015) is a leading example depicts the observed picture of the transactions of industries located in different countries of the world. They are manifestations of the process by which various economic activities are allocated between different countries, i.e. which industries are operated in which countries, on what scales and how their outputs are utilized by industries, households and governments located at home and in other countries of the world. In short WIOT shows the working of the world economic system. Over the years the WIOD has enabled economists and policymakers to understand a large number of international economic phenomena. Some examples are global value chains, fragmentation of international supply chains, transmissions of the impacts of economic events across countries and sectors, identification of hot spots of carbon emissions and other pollutants, changes in the demand for skills due to outsourcing, effects of trade liberalization, estimation of the likely impacts of free trade agreements and so on. The breadth and depth of the applications of the WIOT continues to increase.

In contrast with its widening range of applications, scant attention seems to have been given to the theoretical issues underlying the WIOD, e.g. why are some activities present in some countries but not others, which are the forces determining how the outputs of the industries are absorbed across the world, how are the volumes and values of exports and imports determined, how are the international prices of commodities determined etc. The reasons for this are not hard to find. Firstly, almost all the extant theories of international trade are confined to trade between two countries whereas empirical WIOD's pertain to transactions between several countries. This is obviously true of the Ricardo-Mill and the Heckscher-Ohlin-Samuelson models but it is equally true of the New Trade Theory (Krugman, 1980; Krugman and Helpman, 1985) and the modern revival of Ricardian trade theory (Dornbusch, Fischer and Samuelson, 1977) which allow the two countries to trade in a continuum of consumption commodities. These theories are clearly inadequate for the job of modelling the trade network involving several countries. Secondly, there is as yet no theory that contains a satisfactory treatment of trade in intermediate capital goods, i.e. produced means of production. The Ricardo-Mill model, Graham's (1998) multicountry multicommodity trade model, New Trade theory and the modern revival of Ricardian Trade Theory suppose labour to be the only factor of production, capital goods model are absent. The Heckscher-Ohlin model supposes capital to be a (non-produced) endowment and allows production and trade to take place only in consumption goods. Neo-Ricardian trade theory (Steedman, 1979) and Shiozawa's (2007, 2017) reconstruction of Ricardian trade theory are much more promising in this respect but in their present state of development they are not yet equipped for the task because they contain mathematically indeterminate systems of equations. Thus, extant trade theories have not much to say about the numerous entries in the WIOD which pertain exclusively to the production and trade in purely intermediate capital goods.

The direct consequence of these inadequacies is that applied multi-country trade modelling including Computable General Equilibrium (CGE) and New Quantitative Trade Theory (NQTT) do not consistently conform to any theoretical trade model but borrow from different models depending upon its convenience for empirical treatment in the immediate specific context on hand. Thus, both CGE and NQTT models postulate the existence of a composite capital good which is made up of capital goods whose relative prices are invariant with respect to one another. In so doing they completely ignore the capital theory debates.

Another instance is Eaton and Kortums's (2002) assumption that currency exchange rates are determined by the ratios of money wage rates which they directly borrow from Dornbusch, Fischer and Samuelson (1977) and in doing completely ignore the huge literature on exchange rate determination that emerged during the seventies and thereafter.

These somewhat negative but factual remarks supply the motivation for this paper. The quest is for a theoretical model of international trade that can identify the economic forces that determine the sizes of the intercountry interindustrial transactions. In formal terms the quest is for a theoretical trade model whose unknowns are entries of the world input output table. It will be shown that the required trade model is obtained by appropriately integrating a household demand system with the static open Leontief model [see Mogelli, Neuwahl and Rueda-Cantuche, 2010] Needless to say that the household demand system must have the properties of additivity, homogeneity (of degree zero with respect to prices and incomes) additivity and symmetry (of the substitution matrix). Thus, the Almost Ideal System (AIDS) due to Deaton and Muellbauer (1980) or the Linear Expenditure System (Stone, 1954) are the preferred alternatives. However, for the sake of avoiding purely computational intricacies the simple Cobb-Douglas household demand system has been employed which implies unit income and price elasticities of demand. However, it is emphasized that no aspect of the trade model crucially depends on this assumption. Similar results are obtained if the LES or AIDS demand system are used.

The plan of the paper is as follows. Section 2 outlines the procedure for the determination of equilibrium under autarky. Section 3 presents the two-country two-commodity trade equilibrium. Section 4 presents an example of the determination of international trade equilibrium between three countries in four commodities and displays the corresponding WIOT with entries showing the trade flows between industries and households located across the countries in physical and value terms. In section 5, one of the four commodities is supposed to be non-tradeable commodity. Obviously, it is not internationally traded. The corresponding WIOT has been displayed. Section 5 concludes.

2. Autarkic Equilibrium

Suppose A is an n X n matrix of physical input-output coefficients and L is the n X 1 vector of labour coefficients. Let w be the money wage rate. The prices which equal unit production cost are

$$P = w(I - A^T)^{-1}L \qquad (2.1)$$

Let L_e be the labour endowment. Then the national income at factor cost is wL_e. Let α_i be the propensity to consume commodity i and let $\sum \alpha_i = 1$ implies a static open economy. The quantities demanded for final consumption are

$$C_i = \frac{\alpha_i \, w \, L_e}{p_i} \qquad (2.2)$$

where p_i is obtained from equation (2.1). The gross outputs that must be produced to satisfy the final consumption demands of equation (2) must therefore be

$$X = (I - A)^{-1}C \qquad (2.3)$$

where C and X are nx1 vectors. Clearly X is a vector of market clearing outputs. It can then be shown that (P, X) solution of equations (2.1) and (2.3) automatically ensures the clearing of the labour market i.e. Walras' law holds. To see how, multiply equation (2.3) by wL so that

$$wLX = wL(I - A)^{-1}C$$

Then $$wLX = PC = \frac{p_i \sum \alpha_i \, w \, L_e}{p_i} = wL_e$$

because $\sum \alpha_i = 1$. Therefore, the demand for labour required to produce the gross output vector X i.e. LX is equal to the labour endowment L_e. In effect equations 2.1, 2.2 and 2.3 represents the simplest possible general equilibrium model that is based on input-output analysis.

3. Two-Country Two-Commodity

Consider an example of 2 countries A and B having labour endowments of $L_A = 20,\ L_B = 16$ with $w_A = \$1$ and $w_B = €1$ that spend half of their net national incomes on the two commodities. Table 1 shows the data and their autarky equilibria indicating a trade pattern $A - 1 \quad B - 2$.

Table 1: Autarky Equilibria

$$w_A = \$1 \quad w_B = €1 \quad \propto_{ij} = 0.5 \quad L_A = 20 \quad L_B = 16$$

$$A_A^T = \begin{bmatrix} 0{,}15 & 0.20 \\ 0.17 & 0.25 \end{bmatrix} \ l_A = \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} \ A_B^T = \begin{bmatrix} 0.3 & 0.5 \\ 0.25 & 0.15 \end{bmatrix} \ l_B = \begin{bmatrix} 0.55 \\ 0.35 \end{bmatrix}$$

A B

1. $2.982p_{1A} + 3.977p_{2A} + 8.95w_A = 19.88p_{1A}$ $\qquad 4.744p_{1B} + 7.906p_{2B} + 8.697w_B = 15.81p_{1B}$
2. $3.416p_{1A} + 5.024p_{2A} + 11.05w_A = 20.09p_{2A}$ $\qquad 5.216p_{1B} + 3.130p_{2B} + 7.303w_B = 20.87p_{2B}$

$p_{1A} = \$\,0.742 \qquad p_{1B} = €\,1.367 \qquad p_{2A} = \$\,0.901 \qquad p_{2B} = €\,0.814$

Table 2: Autarky Outputs, Consumption Levels and Prices

C_{iA}	C_{iB}	X_{iA}	X_{iB}	$p_{iA}(\$)$	$p_{iB}(€)$
13.486	5.852	19.885	15.812	0.742	1.367
11.094	9.83	20.094	20.866	0.901	0.814

Observe that $\frac{p_{1A}}{p_{2A}} = 0.823 < 1.679 = \frac{p_{1B}}{p_{2B}}$, indicating that A has the comparative advantage in commodity 1 and B in commodity 2. In exchange rate units the comparative advantage condition can be expressed in the following terms. as $E_{AB}^1 = \frac{p_{1A}}{p_{1B}} = 0.542$ and $E_{AB}^2 = \frac{p_{2A}}{p_{2B}} = 1.107$ gives $p_{1A}p_{2B} < p_{2A}\,p_{1B}$(0.604 <1.231).

If the countries are open to trade the world's economic activity can be depicted as follows,

A B

1. $6.667p_{1A}^T + 8.888p_{2B}^T\,E_{AB} + 20w_A = 44.444p_{1A}^T$ $\qquad --- \, Industry\ 2\ closed \, ---$
2. $--- \, Industry\ 2\ closed \, ---$ $\qquad 11.428p_{1A}^T + 6.857p_{2B}^T\,E_{AB} + 16w_B E_{AB} = 45.714p_{2B}^T E_{AB}$

$$\frac{L_A}{l_{1A}} = \frac{\propto_{1A}\ w_A L_A + \propto_{1B}\ w_B L_B E_{AB}}{p_{1A}^T} + \left[\frac{a_{11A}L_A}{l_{1A}} + \frac{a_{12B}L_B}{l_{2B}}\right] \quad (3.1)$$
$$\frac{L_B}{l_{2B}} = \frac{\propto_{2A}\ w_A L_A + \propto_{2B}\ w_B L_B E_{AB}}{p_{2B}^T E_{AB}} + \left[\frac{a_{21A}L_A}{l_{1A}} + \frac{a_{22B}L_B}{l_{2B}}\right]$$

The international market clearing equation are shown in (3.1). The left-hand sides show the outputs produced and the right-hand sides show the quantities demanded by countries A and B for consumption and intermediate uses. In (3.1) p_{1A}^T, p_{2B}^T denote the post-trade prices and E_{AB} is the equilibrium exchange rate. By Walras' law there is one independent equation in (3.1) to determine 3 unknowns viz prices p_{1A}^T, p_{2B}^T and exchange rate E_{AB}. There is, besides, another difficulty. It is clear from equation (3.1) that E_{AB} can be determined only if either p_{1A}^T or p_{2B}^T, are known. But those prices in turn can be determined only if the exchange rate is known because the price equations expressed in currency A when the countries are open to trade are as shown in (3.2).

$$a_{11A}p_{1A}^T + a_{21A}p_{2B}^T E_{AB} + w_A l_{1A} = p_{1A}^T \quad (3.2)$$
$$a_{12B}p_{1A}^T + a_{22B}p_{2B}^T E_{AB} + w_B l_{2B} E_{AB} = p_{2B}^T E_{AB}$$

There is a circularity among the unknowns of (3.1) and (3.2). To resolve the circularity a system of equations needs to be formed using (say) the first equation of (3.1) and the two equations of (3.2) expressed in currency A to obtain

$$\begin{bmatrix} \frac{L_A - a_{11A}w_A L_A}{l_{1A}} - \frac{a_{12B}w_B L_B}{l_{2B}} & 0 & -a_{12B}w_B L_B \\ 1 - a_{11A} & -a_{21A} & 0 \\ -a_{12B} & 1 - a_{22B} & w_B l_{2B} \end{bmatrix} \begin{bmatrix} p_{1A}^T \\ p_{2B}^T E_{AB} \\ E_{AB} \end{bmatrix} = \begin{bmatrix} a_{11A}w_A L_A \\ w_A L_A \\ 0 \end{bmatrix} \quad (3.3)$$

from which p_{1A}^T, p_{2B}^T and E_{AB} can be simultaneously ascertained. The solution of (3.3) is $E_{AB} = 0.949 \quad p_{1A}^T = \$\ 0.667, \quad p_{2B}^T = €\ 0.619$.

Table 3: With-trade Outputs, Consumption Levels and Prices

	C_{iA}	C_{iB}	Intermediate	X_{iA}	X_{iB}	p_{iA}(\$)	p_{iB}(€)
1	14.981	11.368	18.095	44.444	0	0.668	1.367*
2	17.039	12.929	15.746	0	45.714	0.901*	0.619

*if produce at home

Table 4: WIOT in physical terms

Country	Commodity	1	2	1	2	HC_ij	Import	Export	Output	Intermediate
A	1	6.67	0	0	11.43	14.98	0	22.8	44.44	18.10
	2	0	0	0	0	17.04	25.93	0	0	0
B	1	0	0	0	0	11.37	22.8	0	0	0
	2	8.89	0	0	6.86	12.93	0	25.93	45.71	15.75

Table 5: WIOT in value terms (in Currency A)

Country	Commodity	1	2	1	2	HC_ij	Import	Export	Output	~~Intermediate~~	
A	1	4.45	0	0	7.63	10	0	15.22	29.67	~~12.08~~	~~4.45~~
	2	0	0	0	0	10	15.22	0	0	~~0~~	~~0~~
B	1	0	0	0	0	7.59	15.22	0	0	~~0~~	~~0~~
	2	5.22	0	0	4.02	7.59	0	15.22	26.83	~~9.24~~	~~5.22~~

Observe that gross output of both countries are made up of their component viz domestic intermediate uses, domestic final uses and exports which in turn are equal to foreign intermediate plus final uses e.g. output of country A's =29.67 = domestic intermediate consumption (4.45) + domestic final use (10) +export (7.63+7.59) where country B's intermediate uses 7.63 and final use 7.59. Likewise, for country B.

Observe that the world outputs of both commodities stand at 44.44 and 45.714 units in the with-trade situation, greater than 35.69 and 40.96 units under autarky. Also observe that the levels of consumptions of both commodities in both countries are greater in the with-trade situation than autarky. Further the exchange rate $E_{AB} = 0.9485$ that clears the foreign exchange market (equalises the exports and imports of each country) lies between $E_{AB}^1 = 0.5424$ and $E_{AB}^2 = 1.1076$, i.e. the international terms of trade $\frac{p_{1A}^T}{E_{AB}\, p_{2B}^T} = 1.137$ lies between the domestic terms of trade $\frac{p_{1A}}{p_{2A}} = 0.8226$ and $\frac{p_{1B}}{p_{2B}} = 1.6797$ which shows that buyers in both countries gain by importing instead of producing at home. In short, the international allocation of economic activities A-1 and B-2 in accordance with the principle of comparative advantage enhance the world's outputs and real income.

4. Three-Country Four-Commodity: All Commodities Tradeable

Next consider a more complex situation where 3 countries A, B, and C with labour endowments of $L_j = [500, 600, 800]$ producing 4 commodities and spending 1/4th of their net national incomes for final consumption of commodities in each country. Their input-output and labour coefficients are shown in Table 6.

Table 6: Input-output coefficients

$$A_A^T = \begin{bmatrix} 0.02 & 0.03 & 0.02 & 0.04 \\ 0.10 & 0.08 & 0.06 & 0.09 \\ 0.12 & 0.14 & 0.10 & 0.14 \\ 0.15 & 0.12 & 0.10 & 0.12 \end{bmatrix} \quad l_A = \begin{bmatrix} 0.05 \\ 0.10 \\ 0.15 \\ 0.20 \end{bmatrix}; \quad A_B^T = \begin{bmatrix} 0.10 & 0.08 & 0.07 & 0.12 \\ 0.04 & 0.03 & 0.04 & 0.02 \\ 0.12 & 0.14 & 0.10 & 0.14 \\ 0.15 & 0.12 & 0.10 & 0.10 \end{bmatrix} \quad l_B = \begin{bmatrix} 0.15 \\ 0.05 \\ 0.20 \\ 0.20 \end{bmatrix};$$

$$A_C^T = \begin{bmatrix} 0.10 & 0.12 & 0.09 & 0.10 \\ 0.12 & 0.16 & 0.14 & 0.10 \\ 0.04 & 0.03 & 0.04 & 0.02 \\ 0.02 & 0.03 & 0.05 & 0.03 \end{bmatrix} \quad l_C = \begin{bmatrix} 0.25 \\ 0.20 \\ 0.05 \\ 0.05 \end{bmatrix}$$

$$w_A = \$1 \qquad w_B = €1 \qquad w_C = ¥1$$

Their equilibria under autarky are shown in table 7.

Table 7: Autarky Outputs, Consumption Levels and Prices

i	C_{iA}	C_{iB}	C_{iC}	X_{iA}	X_{iB}	X_{iC}	p_{iA} (\$)	p_{iB}(€)	p_{iC}(¥)
1	1716.984	626.684	597.991	2089.7	1045.032	1026.41	0.073	0.239	0.334
2	776.308	1857.904	650.852	1137.9	2223.553	1145.90	0.161	0.081	0.307
3	507.146	476.826	2593.36	777.52	804.954	3129.09	0.246	0.315	0.077
4	431.558	487.277	2780.814	825.47	855.384	3155.29	0.290	0.308	0.072

From the autarky prices of Table 7 it can be observed that $p_{1A}p_{2B}p_{3C} < p_{1B}p_{2C}p_{3A}$ and $p_{1A}p_{2B}p_{4C} < p_{1B}p_{2C}p_{4A}$ which is McKenzie-Jones lowest-product of-prices criterion or efficient specialisation.[1] This indicates a trade pattern of $A-1 \;\; B-2 \quad C-3{,}4$. The world market clearing equations, corresponding to this trade assignment expressed in a uniform currency A by multiplying the second equation by E_{AB} and third and fourth by E_{AC} are as follows,

$$\frac{L_A}{l_{1A}} = \frac{M_{1A}}{p_{1A}^T} + \left[\frac{a_{11A}L_A}{l_{1A}} + \frac{a_{12B}L_B}{l_{2B}} + \frac{a_{13C}L_{3C}^T}{l_{3C}} + \frac{a_{14C}L_{4C}^T}{l_{4C}}\right]$$

$$\left(\frac{L_B}{l_{2B}}\right) = \frac{M_{2A}}{p_{2B}^T E_{AB}} + \left[\frac{a_{21A}L_A}{l_{1A}} + \frac{a_{22B}L_B}{l_{2B}} + \frac{a_{23C}L_{3C}^T}{l_{3C}} + \frac{a_{24C}L_{4C}^T}{l_{4C}}\right] \qquad (4.1a)$$

$$\left(\frac{L_{3C}^T}{l_{3C}}\right) = \frac{M_{3A}}{p_{3C}^T E_{Ac}} + \left[\frac{a_{31A}L_A}{l_{1A}} + \frac{a_{32B}L_B}{l_{2B}} + \frac{a_{33C}L_{3C}^T}{l_{3C}} + \frac{a_{34C}L_{4C}^T}{l_{4C}}\right]$$

$$\left(\frac{L_{4C}^T}{l_{4C}}\right) = \frac{M_{4A}}{p_{4C}^T E_{Ac}} + \left[\frac{a_{41A}L_A}{l_{1A}} + \frac{a_{42B}L_B}{l_{2B}} + \frac{a_{43C}L_{3C}^T}{l_{3C}} + \frac{a_{44C}L_{4C}^T}{l_{4C}}\right]$$

[1] The McKenzie -Jones condition [McKenzie, 1954; Jones, 1961] are used to identify efficient specializations. As originally formulated they are applicable to situations in which number of countries equal the number of commodities. However, nothing prevents its sequential application to the data of autarky prices until every country has been assigned to at least one country. Thus, for the example whose data are shown in table 7 the lowest product of prices is obtained by choosing the smallest price of a commodity in each country; $p_{1A} = 0.073, p_{2B} = 0.081$ and $p_{4C} = 0.072$ are the lowest prices of commodities 1, 2 and 4 in countries A, B and C. Any assignment of 1, 2 and 4 to countries other A, B and C would be inefficient assignment. But this leaves commodity 3. The next lowest product of price that includes commodity 3 is $p_{1A}\, p_{2B}p_{4C}$ which indicate an efficient assignment A-1, B-2, C-3. This completes the initial assignment of the commodities to the countries. It is important to add a caveat. In our example we have supposed wage rates equal to one unit of currency in each country. If wage rates that are different from unity are chosen the money prices will need to be normalised by the money wage, i.e. products of p_{ij}/w_j (i=1,2,….n; j=A,B,…Z) must be considered.

where $M_{iA} = \sum \propto_{ij} w_j L_j E_{Aj}$ $(i = 1, \ldots 4, j = A, B, C)$ expresses the final consumption expenditures on the four commodities in terms of currency A. Full employment equation of C is as follows

$$L_C = L_{3C}^T + L_{4C}^T \qquad (4.1b)$$

Equations (4.1a) and (4.1b) contain 5 independent equations in 4 unknowns; viz. $E_{AB}, E_{AC}, L_{3C}^T, L_{4C}^T$. There is however difficulty that must be overcome[2]. It arises due to the fact that the equations of (4.1a) are non-linear because of the presence of products of unknown prices and exchange rates in the denominators of the equation (4.1a). There is, moreover, the presence of unknown prices of the commodities p_{ij}^T in the denominators on the right-hand side of (4.1a). To overcome the problems of non-linearity recourse needs to be taken to a process of trial and error. Consider the with-trade world price system,

$$\begin{aligned}
&a_{11A}p_{1A}^T + a_{21A}p_{2B}^T E_{AB} + a_{31A}p_{3C}^T E_{AC} + a_{41A}p_{4C}^T E_{AC} + w_A l_{1A} = p_{1A}^T \\
&a_{12B}p_{1A}^T + a_{22B}p_{2B}^T E_{AB} + a_{32B}p_{3C}^T E_{AC} + a_{42B}p_{4C}^T E_{AC} + w_B l_{2B} E_{AB} = p_{2B}^T E_{AB} \qquad (4.2) \\
&a_{13C}p_{1A}^T + a_{23C}p_{2B}^T E_{AB} + a_{33C}p_{3C}^T E_{AC} + a_{43C}p_{4C}^T E_{AC} + w_C l_{3C} E_{AC} = p_{3C}^T E_{AC} \\
&a_{14C}p_{1A}^T + a_{24C}p_{2B}^T E_{AB} + a_{34C}p_{3C}^T E_{AC} + a_{44C}p_{4C}^T E_{AC} + w_C l_{4C} E_{AC} = p_{4C}^T E_{AC}
\end{aligned}$$

in which all prices are expressed in currency A. Initially autarky price used to get exchange rate by equation (4.1a).

$$\begin{bmatrix} 1 - a_{11A} & -a_{21A} E_{AB} & -a_{31A} E_{AC} & 0 \\ -a_{12B} & (1 - a_{22B}) E_{AB} & -a_{32B} E_{AC} & 0 \\ -a_{13C} & -a_{23C} E_{AB} & (1 - a_{33C}) E_{AC} & a_{43C} E_{AC} \\ -a_{14C} & -a_{24C} E_{AB} & -a_{34C} E_{AC} & (1 - a_{44C}) E_{AC} \end{bmatrix} \begin{bmatrix} p_{1A}^T \\ p_{2B}^T \\ p_{3C}^T \\ p_{4C}^T \end{bmatrix} = \begin{bmatrix} w_A l_{1A} \\ w_B l_{2B} E_{AB} \\ w_C l_{3C} E_{AC} \\ w_C l_{4C} E_{AC} \end{bmatrix}$$

Equations (4.2) can be used alternately with (4.1) to find the solution. Thus, initially substitute say the autarky prices p_{ij} in (4.1a) to obtain a tentative solution for $L_{3C}^T, L_{4C}^T, E_{AB}, E_{AC}$. Next substitute E_{AB}, E_{AC} thus obtained in (4.2) to obtain a tentative solution for p_{ij}^T which may be substituted back in

[2] With Z countries (A, B, …Z) there are z(z-1) exchange rates, of which currency against every other currency because $E_{ii} = 1$. Therefore, in our example of 3 countries there are actually 3X2=6 exchange rates to be determined. However, arbitrage forces operating in the foreign exchange markets ensure that

$E_{ij} = \frac{1}{E_{ji}}$ $\forall\, i, j = A, B, C \;\; i \neq j$ and $E_{ij} E_{ik} = E_{ik}$ $\forall\, i, j = A, B, C \;\; i \neq j \neq k$

These conditions eliminate between there exchange rates viz E_{BA}, E_{CA}, E_{BC}, and E_{CB} leaving in effect 2 exchange rates E_{AB} and E_{AC} to be determined by the international market clearing and domestic full employment equations.

(4.1), repeating the process until convergence to the desired accuracy level is obtained. For the problem on hand a solution up to an accuracy of five places of decimals is found in 7 iterations. The solution is obtained is shown in Table 8.

Table 8: With-trade Outputs, Consumption Levels, and Prices

	C_{iA}	C_{iB}	C_{iC}	Intermediate	X_{iA}	X_{iB}	X_{iC}	p_{iA}($)	p_{iB}(€)	p_{iC}(¥)
1	2193.392	2013.341	4629.352	1163.915	10000	0	0	0.057	0	0
2	2695.791	2474.499	5689.710	1140.000	0	12000	0	0	0.061	0
3	1687.407	1548.891	3561.425	1398.042	0	0	8195.77	0	0	0.056
4	1679.580	1541.706	3544.904	1038.042	0	0	7804.23	0	0	0.056

$$L_{3C}^{T} = 409.788 \qquad L_{4C}^{T} = 390.212 \qquad E_{AB} = 0.765 \qquad E_{AC} = 1.319$$

The entries in table 8 may be duly compared with those obtained under autarky shown in table 7 to observe the advantage from trade. The world output vector (10000, 12000, 8195.77, 7804.23) clearly dominates the output vector obtaining under autarky (416.14, 4507.353, 4175.83, 4461.66). Moreover, the levels of consumption of all 4 commodities in the 3 countries stand are greater than under autarky. The commodity prices in the with-trade situation are lower than those under autarky. The world input-output matrices in physical and value terms corresponding to the world trade equilibrium are shown in tables 9 and 10.

Country A

1. $$200p_{1A}^{T} + 300p_{2B}^{T}E_{AB} + 200p_{3C}^{T}E_{AC} + 400p_{4C}^{T}E_{AC} + 500\, w_{A} = 10000\, p_{1A}^{T}$$

---------*Industry 2, 3 4 closed*------------------

Country B

2. $$480p_{1A}^{T} + 360p_{2B}^{T}E_{AB} + 480p_{3C}^{T}E_{AC} + 240p_{4C}^{T}E_{AC} + 600\, w_{B}\, E_{AB} = 12000p_{2B}^{T}E_{AB}$$

---------*Industry 1, 3 4 closed*------------------ (4.3)

Country C

---------*Industry 1 and 2 closed*------------------

3. $$327.83p_{1A}^{T} + 245.87p_{2B}^{T}E_{AB} + 327.83p_{3C}^{T}E_{AC} + 163.92p_{4C}^{T}E_{AC} + 409.79w_{C}\, E_{AC} = 8195.76p_{3C}^{T}E_{AC}$$

4. $$156.08p_{1A}^{T} + 234.13p_{2B}^{T}E_{AB} + 390.21p_{3C}^{T}E_{AC} + 234.13p_{4C}^{T}sE_{AC} + 390.21w_{C}E_{AC} = 7804.23p_{4C}^{T}E_{AC}$$

Table 9: WIOT in physical terms

Country		A				B				C							
		1	2	3	4	1	2	3	4	1	2	3	4	HC_ij	Import	Export	Output
A	1	200	0	0	0	0	480	0	0	0	0	327.83	156.08	2193.39	0	7606.61	10000
	2	0	0	0	0	0	0	0	0	0	0	0	0	2695.79	2995.79	0	0
	3	0	0	0	0	0	0	0	0	0	0	0	0	1687.41	1887.41	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	1679.58	2079.58	0	0
B	1	0	0	0	0	0	0	0	0	0	0	0	0	2013.34	2493.34	0	0
	2	300	0	0	0	0	360	0	0	0	0	245.87	234.13	2474.5	0	9165.5	12000
	3	0	0	0	0	0	0	0	0	0	0	0	0	1548.89	2028.89	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	1541.71	1781.71	0	0
C	1	0	0	0	0	0	0	0	0	0	0	0	0	4629.35	5113.27	0	0
	2	0	0	0	0	0	0	0	0	0	0	0	0	5689.71	6169.71	0	0
	3	200	0	0	0	0	480	0	0	0	0	327.83	390.21	3561.42	0	3916.3	8195.76
	4	400	0	0	0	0	240	0	0	0	0	163.92	234.13	3544.9	0	3861.29	7804.23

Table 10: WIOT in value terms (in Currency A)

Country		A				B				C							
		1	2	3	4	1	2	3	4	1	2	3	4	HC_ij	Import	Export	Output
A	A	11.4	0	0	0	0	27.35	0	0	0	0	18.68	8.9	125	0	433.5	569.89
	2	0	0	0	0	0	0	0	0	0	0	0	0	125	138.91	0	0
	3	0	0	0	0	0	0	0	0	0	0	0	0	125	139.82	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	125	154.77	0	0
B	B	0	0	0	0	0	0	0	0	0	0	0	0	114.74	142.09	0	0
	2	13.91	0	0	0	0	16.69	0	0	0	0	11.4	10.86	114.74	0	424.99	556.42
	3	0	0	0	0	0	0	0	0	0	0	0	0	114.74	150.3	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	114.74	132.6	0	0
C	C	0	0	0	0	0	0	0	0	0	0	0	0	263.82	291.4	0	0
	2	0	0	0	0	0	0	0	0	0	0	0	0	263.82	286.08	0	0
	3	14.82	0	0	0	0	35.56	0	0	0	0	24.29	28.91	263.82	0	290.11	607.13
	4	29.77	0	0	0	0	17.86	0	0	0	0	12.2	17.42	263.82	0	287.37	580.82

5. Three-Country Four-Commodity:One Commodity Non-tradeable

Consider the example of section 4 and its autarky solution. But suppose commodity 4 to be non-tradeable. Applying the McKenzie-Jones product of prices criterion to tradeable commodities shows

$$p_{1A}p_{2B}p_{3C} < p_{1B}p_{2C}p_{3A} < p_{2B}p_{1C}p_{3A} \ldots ..$$

Indicating A-1, B-2, C-3 as the trade assignment that is a promising candidate for the equilibrium assignment. Hence, the trade pattern that presents itself for trial is $A-1, B-2, C-3$. The world market clearing equations, corresponding to this trade assignment expressed in a uniform currency are as follows,

$$\frac{L_{1A}}{l_{1A}} = \frac{M_{1A}}{p_{1A}^T} + \left(\frac{a_{11A}L_{1A}}{l_{1A}} + \frac{a_{12B}L_{2B}}{l_{2B}} + \frac{a_{13C}L_{3C}}{l_{3C}} + \frac{a_{14A}L_{4A}}{l_{4A}} + \frac{a_{14B}L_{4B}}{l_{4B}} + \frac{a_{14C}L_{4C}}{l_{4C}}\right)$$

$$\frac{L_{2B}}{l_{2B}} = \frac{M_{2A}}{p_{2B}^T E_{AB}} + \left(\frac{a_{21A}L_{1A}}{l_{1A}} + \frac{a_{22B}L_{2B}}{l_{2B}} + \frac{a_{23C}L_{3C}}{l_{3C}} + \frac{a_{24B}L_{4B}}{l_{4B}} + \frac{a_{24B}L_{4B}}{l_{4B}} + \frac{a_{24C}L_{4C}}{l_{4C}}\right)$$

$$\frac{L_{3C}}{l_{3C}} = \frac{M_{3A}}{p_{2C}^T E_{AC}} + \left(\frac{a_{31A}L_{1A}}{l_{1A}} + \frac{a_{32B}L_{2B}}{l_{2B}} + \frac{a_{33C}L_{3C}}{l_{3C}} + \frac{a_{34C}L_{4C}}{l_{4C}} + \frac{a_{34B}L_{4B}}{l_{4B}} + \frac{a_{34C}L_{4C}}{l_{4C}}\right)$$

$$\frac{L_{4A}}{l_{4A}} = \frac{\alpha_4\, w_A\, L_A}{p_{4A}^T} + \left(\frac{a_{41A}L_{1A}}{l_{1A}} + \frac{a_{44A}L_{4A}}{l_{4A}}\right)$$

$$\frac{L_{4B}}{l_{4B}} = \frac{\alpha_4\, w_B L_B}{p_{4B}^T} + \left(\frac{a_{42B}L_{2B}}{l_{2B}} + \frac{a_{44B}L_{4B}}{l_{4B}}\right) \qquad (5.1)$$

$$\frac{L_{4C}}{l_{4C}} = \frac{\alpha_4\, w_C\, L_C}{p_{4C}^T} + \left(\frac{a_{43C}L_{3C}}{l_{3C}} + \frac{a_{44C}L_{4C}}{l_{4C}}\right)$$

$$L_A = L_{1A} + L_{4A}\,; \qquad L_B = L_{2B} + L_{4B}; \qquad L_C = L_{3C} + L_{4C}$$

where $M_{iA} = \sum \propto_{ij} w_j L_j E_{Aj}$ $(i = 1, \ldots 4, j = A, B, C)$ expresses the final consumption expenditures on the four commodities in terms of currency A. Equations (5.1) contain 9 independent equations in 8 unknowns; viz. $E_{AB}, E_{AC}, L_{1A}^T, L_{4A}^T, L_{2B}^T, L_{4B}^T, L_{3C}^T, L_{4C}^T$.

Post-trade world price system is as follows,

$$a_{11A}\, p_{1A}^T + a_{21A}\, p_{2B}^T E_{AB} + a_{31A}\, p_{3C}^T E_{AC} + a_{41A}\, p_{4A}^T + w_A l_{1A} = p_{1A}^T$$

$$a_{14A}\, p_{1A}^T + a_{24A}\, p_{2B}^T E_{AB} + a_{34A}\, p_{3C}^T E_{AC} + a_{44A}\, p_{4A}^T + w_A l_{4A} = p_{4A}^T$$

$$a_{12B}\, p_{1A}^T + a_{22B}\, p_{2B}^T E_{AB} + a_{32B}\, p_{3C}^T E_{AC} + a_{42B}\, p_{4B}^T E_{AB} + w_B l_{2B} E_{AB} = p_{2B}^T E_{AB}$$

$$a_{14B}\, p_{1A}^T + a_{24B}\, p_{2B}^T E_{AB} + a_{34B}\, p_{3C}^T E_{AC} + a_{44B}\, p_{4B}^T E_{AB} + w_B l_{4B} E_{AB} = p_{4B}^T E_{AB} \qquad (5.2)$$

$$a_{13C}\, p_{1A}^T + a_{23C}\, p_{2B}^T E_{AB} + a_{33C}\, p_{3C}^T E_{AC} + a_{43C}\, p_{4C}^T E_{AC} + w_C l_{3C} E_{AC} = p_{3C}^T E_{AC}$$

$$a_{14C}\, p_{1A}^T + a_{24C}\, p_{2B}^T E_{AB} + a_{34C}\, p_{3C}^T E_{AC} + a_{44C}\, p_{4C}^T E_{AC} + w_C l_{4C} E_{AC} = p_{4C}^T E_{AC}$$

in which all prices are expressed in currency A.

$$\begin{bmatrix} 1-a_{11A} & -a_{21A}E_{AB} & -a_{31A}E_{AC} & -a_{41A} & 0 & 0 \\ -a_{14A} & -a_{24A}E_{AB} & -a_{34A}E_{AC} & (1-a_{44A}) & 0 & 0 \\ -a_{12B} & (1-a_{22B})E_{AB} & -a_{32B}E_{AC} & 0 & -a_{42B}E_{AB} & 0 \\ -a_{14B} & -a_{24B}E_{AB} & -a_{34B}E_{AC} & 0 & (1-a_{44B})E_{AB} & 0 \\ -a_{13C} & -a_{23C}E_{AB} & (1-a_{33C})E_{AC} & 0 & 0 & a_{43C}E_{AC} \\ -a_{14C} & -a_{24C}E_{AB} & -a_{34C}E_{AC} & 0 & 0 & (1-a_{44C})E_{AC} \end{bmatrix} \begin{bmatrix} p_{1A}^T \\ p_{2B}^T \\ p_{3C}^T \\ p_{4A}^T \\ p_{4B}^T \\ p_{4C}^T \end{bmatrix} = \begin{bmatrix} w_A l_{1A} \\ w_A l_{4A} \\ w_B l_{2B}E_{AB} \\ w_B l_{4B}E_{AB} \\ w_C l_{3C}E_{AC} \\ w_C l_{4C}E_{AC} \end{bmatrix}$$

As before equations (5.1) and (5.2) can be solved iteratively by supposing that p_{ij}^T are equal to autarky prices initially. Obtaining a tentative solution for the exchange rates from (5.1). Substituting it in (5.2) to obtain a better approximation of p_{ij}^T and repeating the process until full convergence. For the problem on hand a solution up to an accuracy of five places of decimals is found in 7 iterations. The solution is obtained as shown in Table 11.

Table 11: With-trade Consumption Levels, Outputs, and Prices: One Commodity Non-tradeable

i	C_{iA}	C_{iB}	C_{iC}	Intermediate	X_{iA}	X_{iB}	X_{iC}	p_{iA}($)	p_{iB}(€)	p_{iC}(¥)
1	1981.366	1696.664	1540.103	1300.646	6518.80	0	0	0.063	0	0
2	2821.962	2416.475	2193.492	1138.452	0	8570.381	0	0	0.062	0
3	4177.127	3576.918	3246.854	1414.035	0	0	12414.90	0	0	0.062
4	505.118	600.257	3321.208	886.193	870.31	857.405	3585.07	0.247	0.250	0.060

$$E_{AB} = 0.714 \qquad E_{AC} = 0.486 \qquad L_{1A}^T = 325.939 \qquad L_{4A}^T = 174.061 \qquad L_{2B}^T = 428.519$$

$$L_{4B}^T = 171.480 \quad L_{3C}^T = 620.747 \quad L_{4C}^T = 179.253$$

A comparison of table 11 with table 7 shows the gains of trade; world outputs and levels of consumption are greater than those in autarky for all commodities including the non-tradeable commodity 4.

Country A

1. $200p_{1A}^T + 300p_{2B}^T E_{AB} + 200p_{3C}^T E_{AC} + 400p_{4C}^T E_{AC} + 500\, w_A = 10000\, p_{1A}^T$

---------*Industry 2, 3 closed*------------------

Country B

2. $480p_{1A}^T + 360p_{2B}^T E_{AB} + 480p_{3C}^T E_{AC} + 240p_{4C}^T E_{AC} + 600\, w_B\, E_{AB} = 12000p_{2B}^T E_{AB}$

---------*Industry 1, 3 closed*------------------

(4.3)

Country C

---------*Industry 1 and 2 closed*------------------

3. $327.83p_{1A}^T + 245.87p_{2B}^T E_{AB} + 327.83p_{3C}^T E_{AC} + 163.92p_{4C}^T E_{AC} + 409.79w_C\, E_{AC} = 8195.76p_{3C}^T E_{AC}$
4. $156.08p_{1A}^T + 234.13p_{2B}^T E_{AB} + 390.21p_{3C}^T E_{AC} + 234.13p_{4C}^T s E_{AC} + 390.21w_C E_{AC} = 7804.23p_{4C}^T E_{AC}$

Table 12 and 13 show the world input-output matrices in physical and value terms.

Table 12 WIOT in physical terms

Country		A				B				C							
		1	2	3	4	1	2	3	4	1	2	3	4	HC_ij	Import	Export	Output
A	1	130.38	0	0	130.55	0	342.82	0	128.61	0	0	496.6	71.7	1981.37	0	4276.49	6518.78
	2	0	0	0	0	0	0	0	0	0	0	0	0	2821.96	3121.96	0	0
	3	0	0	0	0	0	0	0	0	0	0	0	0	4177.13	4394.53	0	0
	4	260.75	0	0	104.44	0	0	0	0	0	0	0	0	505.12	0	0	870.31
B	1	0	0	0	0	0	0	0	0	0	0	0	0	1696.66	2168.09	0	0
	2	195.56	0	0	104.44	0	257.11	0	102.89	0	0	372.45	107.55	2416.48	0	5795.45	8570.38
	3	0	0	0	0	0	0	0	0	0	0	0	0	3576.92	4005.47	0	0
	4	0	0	0	0	0	171.41	0	85.74	0	0	0	0	600.26	0	0	857.4
C	1	0	0	0	0	0	0	0	0	0	0	0	0	1540.1	2108.4	0	0
	2	0	0	0	0	0	0	0	0	0	0	0	0	2193.49	2673.49	0	0
	3	130.38	0	0	87.03	0	342.82	0	85.74	0	0	496.6	179.25	3246.85	0	8400.01	12414.93
	4	0	0	0	0	0	0	0	0	0	0	248.3	107.55	3321.21	0	0	3585.07

Table 13 WIOT in value term (in Currency A)

Country		A				B				C				HC_ij	Import	Export	Output
		1	2	3	4	1	2	3	4	1	2	3	4				
A	1	8.23	0	0	8.24	0	21.63	0	8.11	0	0	31.33	4.52	125	0	269.79	411.26
	2	0	0	0	0	0	0	0	0	0	0	0	0	125	138.29	0	0
	3	0	0	0	0	0	0	0	0	0	0	0	0	125	131.51	0	0
	4	64.53	0	0	25.84	0	0	0	0	0	0	0	0	125	0	0	215.37
B	1	0	0	0	0	0	0	0	0	0	0	0	0	107.04	136.78	0	0
	2	8.66	0	0	4.63	0	11.39	0	4.56	0	0	16.5	4.76	107.04	0	256.71	379.63
	3	0	0	0	0	0	0	0	0	0	0	0	0	107.04	119.86	0	0
	4	0	0	0	0	0	30.57	0	15.29	0	0	0	0	107.04	0	0	152.89
C	1	0	0	0	0	0	0	0	0	0	0	0	0	97.16	133.01	0	0
	2	0	0	0	0	0	0	0	0	0	0	0	0	97.16	118.42	0	0
	3	3.9	0	0	2.6	0	10.26	0	2.57	0	0	14.86	5.36	97.16	0	251.37	371.52
	4	0	0	0	0	0	0	0	0	0	0	7.26	3.15	97.16	0	0	104.88

The WIOT results reveal that trade leads to clear gains in consumption and efficiency, driven by specialization according to comparative advantage. Production becomes internationally fragmented, forming global value chains where intermediate goods cross borders. While some sectors remain domestically produced, tradable sectors exhibit strong specialization patterns. These dynamics increase interdependence across countries and generate measurable welfare gains relative to autarky.

6. Concluding remarks

The principal purpose of this paper has been to formulate a theoretical trade model whose solution takes the form of a world input output table, i.e. its entries show the intercountry interindustrial transactions in physical and value terms. This objective, we think, has been achieved. The paper has formulated an international trade model that is based on the static open Leontief system that is augmented by a household final consumption demand system to determine consumption sizes endogenously. The results of the trade model depend upon three determinants a) the pattern of comparative advantages b) the sizes of the countries in terms of their labour endowments and c) the propensities to consume the different goods in the countries. Changes in these determinants will cause changes in the international trade equilibrium including changes in the trade assignments as well. Having said that it must be admitted that the model is elementary. It will need to be generalized in several directions to enhance its relevance for real world trade. These directions include making room for (i) saving, investment and economic growth (ii) taxation, tariffs and government expenditures (iii) differentiated commodities and intraindustry trade (iv) the distribution of incomes between wages and profits and (v) integrating environmental accounts if possible.

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