Combining Agent-Based and Input–Output Modeling in Economics:

The Multi-Agent Input–Output (MAIO) Framework

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Abstract

We introduce MAIO2, the second term of a sequence of multi-sectoral agent-based macroeconomic models initiated by [1]. Building on the first term, the model adds fixed capital and labour as complementary production factors, a single bank with an endogenous credit channel, a government with sectoral taxation and lump-sum transfers, and a foreign sector with exogenous demand. The three new flows—of goods, labour, and credit—are exchanged through decentralised matching protocols of varying sophistication: a preferred-supplier cascade with rationing for goods, a random meeting protocol for labour, and a single-bank binary granting rule for credit. We derive a closed-form steady state for a benchmark configuration with one KAU per sector, a stationary foreign sector, a fully accommodating credit regime, and a constant fiscal stance; the production system retains a linear input–output form, solvable via Perron–Frobenius arguments, and heterogeneous households are absorbed through group-level income shares. We extend the indistinguishability conditions of MAIO1 to the institutional channels of MAIO2 and identify strong relational indistinguishability as the binding condition for exact aggregation, with bank dividends, government transfers, and foreign trade entering the macroscopic agent through aggregation-compatible weights by construction. The framework supports SAM-based calibration, with applications to the Italian energy transition described in companion work.

Keywords: Agent-based modeling; Input–output analysis; Multi-Agent Input–Output model; Heterogeneous households; Stock-flow consistency; Structural change.
JEL classification: C63, E12, E16, E27.

1 Introduction

The methodological programme. This paper extends the multi-agent input–output (MAIO) framework introduced in [1], henceforth MAIO1. That paper proposed a sequence of multi-sectoral, multi-agent models designed to satisfy four formal properties: explicit heterogeneity and decentralised interaction; qualitative tractability of at least one term; explicit conditions under which aggregate dynamics can be represented without reference to complex factors; and sequential comparability across terms. The first term of the sequence was deliberately stripped of behavioural detail—fixed prices, naive expectations, no inventory targets, no supplier switching beyond rationing—in order to obtain a partially analytically tractable benchmark. The present paper develops the second term, which we call MAIO2, in which several of those restrictions are relaxed and the institutional structure of the economy is closed.

What MAIO2 adds. MAIO2 differs from the first term along five dimensions, each motivated by the methodological criterion of progressive enrichment. First, production no longer rests on intermediate inputs alone: it requires fixed *capital* and *labour* as complementary factors, with sector-specific intensities. Second, capital adjustment follows a Sraffian-supermultiplier accelerator rule, anchored to expected production and a target capital utilisation rate [8]. Third, prices become *endogenous*: each producer sets a markup over the unit cost of production, with the markup itself responsive to demand pressure. Fourth, two new institutional agents close the economy: a *government* that levies taxes and consumes sectoral output, and a *rest of the world* that absorbs domestic exports and supplies imports. Fifth, a *bank* intermediates the credit relationship between firms and households, holding deposits and granting loans according to a parametric policy. Together, these elements convert the model from an open production system into a stock-flow consistent multi-agent macroeconomy [9].

The decentralised structure as a load-bearing assumption. A central methodological commitment, inherited from MAIO1, is that exchange remains decentralised. In MAIO1 this was confined to the goods market, where a preferred-supplier protocol with cascading quantity rationing and threshold-based switching generates an endogenous purchasing network responsible for path dependence and network reconfiguration. In MAIO2 we retain that protocol for the goods market (Section 3.1) and complement it with two further decentralised markets, each governed by its own matching protocol. The labour market (Section 3.2) is matched through a random meeting protocol that produces an endogenous, period-by-period employment graph between households and KAUs, with rationing emerging on both sides as unemployment and unfilled vacancies. The credit market (Section 3.3) operates through a single-bank, binary-granting protocol that links every firm to the unique financial intermediary, with rationing reduced to the two-state alternative of full granting and credit denial. The three markets thus share two structural features—decentralised matching and the possibility of rationing—while differing in the sophistication of the matching mechanism. By imposing decentralised exchange on the three markets in which goods, labour, and credit are exchanged, MAIO2 inherits and extends the methodological commitment of the first term: multiplicity and interaction across markets are not a description of applications, but a structural feature of the model.

Analytical contribution. The analytical contribution of the paper is twofold. First, we derive a closed-form steady state for a benchmark version of MAIO2 in which (i) each sector contains exactly one local kind-of-activity unit, (ii) households are heterogeneous (multiple groups with distinct propensities, ownership shares, and consumption profiles), (iii) prices are constant, (iv) the rest of the world supplies a constant aggregate income, and (v) no rationing occurs in any of the three markets. The benchmark generalises the steady state of MAIO1 to an environment with capital, labour, government, and trade, and shows how the conserved quantity $\mathcal{K}_0$ of the first term is modified by the new institutional flows. Second, we extend the indistinguishability conditions P1–P3 of MAIO1 to MAIO2: we identify the symmetries on the new parameters (markups, capital and labour intensities, target capital utilisation rates, tax and transfer schedules, foreign demand structure) under which exact aggregation is preserved.

Related literature. MAIO2 is a continuation of the modelling tradition initiated with the Eurace family [2,3] and shares with the Schumpeter–Keynes models of Dosi–Lamperti–Roventini [4] a commitment to multi-sector, agent-based, stock-flow consistent macro modelling. Relative to recent data-driven AB-IO models [5,7], MAIO2 emphasises analytical tractability over fit, in the spirit of the methodological programme described in MAIO1. The supermultiplier accelerator employed for capital adjustment follows the Sraffian tradition revived by [8]; the SFC discipline imposed on monetary flows follows [9].

Outline. Section 2 describes the agents and their state and decision rules. Section 3 formalises the three decentralised markets and the bilateral networks they generate. Section 4 fixes the period timing and lists the behavioural restrictions adopted in this paper. Section 5 derives the steady state of the benchmark and discusses existence and uniqueness. Section 6 extends the indistinguishability conditions to MAIO2. Section 7 concludes and outlines the next term of the sequence.

2 The model

The economy of MAIO2 is populated by six types of agent: households, non-financial firms, local kind-of-activity units (KAUs), a bank, a government, and a rest-of-the-world (RoW) sector. Households and firms inherit their structure from MAIO1; the local KAUs inherit the input–output backbone but acquire two new factors (capital and labour) and an endogenous pricing rule; the bank, the government and the RoW are introduced for the first time in this term of the sequence. We follow the expository structure of MAIO1: each agent is described by a triple of (i) state, (ii) decision rules, and (iii) interactions. The decentralised protocols that link these agents through goods, labour, and credit markets are formalised in Section 3.

We index time by t. Sectors are indexed by $i = 1, \ldots, N_A$; KAUs by k; firms by f; households by $h = 1, \ldots, N_H$; the bank, government, and RoW are unique agents. Whenever notation coincides with MAIO1 it carries the same meaning; new symbols are introduced explicitly at first use.

2.1 Households

Households are the ultimate recipients of dividends, the holders of the stock of money, and the demanders of final consumption goods. Relative to MAIO1, the only structural change is the relaxation of restriction A1: the model now admits $N_H \geq 1$ heterogeneous households.

State. At time t, each household h is characterised by:

- the money stock $M_{h,t} \in \mathbb{R}_{\geq 0}$, which is the only liquid component of wealth (ownership shares are non-marketable, exactly as in MAIO1);
- a vector of ownership shares $(S_{hf})_{f=1,\ldots,N_F}$ over non-financial firms, with $\sum_h S_{hf} = 1$ for every f;
- a vector of ownership shares $(S_{hb})_{b\in\mathcal{B}}$ over banks, with $\sum_h S_{hb} = 1$ for every b (Section 2.4);
- the identity of a *primary bank* $b(h) \in \mathcal{B}$, at which the household holds its deposit account and through which all money flows of period t are settled (Section 2.4);
- the realised income $Y_{h,t}$, accumulated over the period from firms' and banks' dividends and—when the labour market is active—wages;
- a fixed vector of expenditure shares $\boldsymbol{\alpha}_h = (\alpha_{1h}, \ldots, \alpha_{N_A h})^\top$ with $\sum_i \alpha_{ih} = 1$.

The constant individual parameters are: the propensity to consume out of income c_h^y, the propensity to consume out of wealth c_h^w, and the target wealth-to-income ratio λ_h^T. Following MAIO1, these are related to a single buffer-stock parameter ξ_h via $c_h^y = 1 - \xi_h \lambda_h^T$ and $c_h^w = \xi_h$.

Decision rules. Income $Y_{h,t}$ comprises dividends from owned firms and banks and, when the labour market is active, wages from KAUs that employ household members. Letting $\pi_{k,t-1}$ denote the profit of KAU k in the previous period and Ω_{fk} the firm-to-KAU quota matrix introduced in Section 2.2, dividend income from firms reads

$$Y_{h,t}^{\mathrm{div},F} \;=\; \sum_{f,k} S_{hf}\, \Omega_{fk}\, \pi_{k,t-1}. \tag{1}$$

Dividend income from banks reads

$$Y_{h,t}^{\mathrm{div},B} \;=\; \sum_{b\in\mathcal{B}} S_{hb}\, \Pi_{b,t-1}^{B}, \tag{2}$$

with $\Pi_{b,t-1}^B$ the period-$(t-1)$ profit of bank b defined in Section 2.4. The two expressions are the analogues of equation (16) in MAIO1, with banks added as a parallel ownership channel; the only difference is that the household subscript h is no longer redundant. Wage income is specified once the labour market protocol is in place (Section 3.2). Aggregate household income is $Y_{h,t} = Y_{h,t}^{\mathrm{div},F} + Y_{h,t}^{\mathrm{div},B} + W_{h,t} + T_{h,t}$, where $W_{h,t}$ are wages received and $T_{h,t}$ are net transfers from the government (positive transfers minus taxes paid; see Section 2.5).

The consumption budget follows the buffer-stock rule of MAIO1: each household targets a wealth-to-income ratio λ_h^T and the period budget reacts to deviations,

$$B_{h,t} \;=\; Y_{h,t} \;+\; \xi_h\big(M_{h,t-1} - \lambda_h^T\, Y_{h,t}\big) \;=\; c_h^y\, Y_{h,t} + c_h^w\, M_{h,t-1}. \tag{3}$$

The total budget is allocated across sectoral goods according to fixed expenditure shares,

$$B_{ih,t} \;=\; \alpha_{ih}\, B_{h,t}. \tag{4}$$

Within each sector, $B_{ih,t}$ is converted into a planned physical demand through the preferred-supplier price observed by household h (Section 3.1); rationing in the goods market may leave part of $B_{ih,t}$ unspent.

Interactions. Households interact with two other agent types. With *firms*, through the ownership share matrix S that drives equation (1). With local *KAUs*, through the goods market: each household carries an ordered list of preferred suppliers for each product type and follows the decentralised protocol of Section 3.1. When the labour market is active (Section 3.2), households interact with KAUs also as workers, and with the government through the tax/transfer channel of Section 2.5. The law of motion of household money is the identity

$$M_{h,t} \;=\; M_{h,t-1} \;+\; Y_{h,t} \;-\; f_{h,t}, \tag{5}$$

where $f_{h,t} = \sum_i f_{ih,t}$ is the realised consumption expenditure of h. Unspent budget $(B_{ih,t} - f_{ih,t})$ is possible if the goods market clears with rationing.

Remark. Equations (1)–(5) reduce to those of MAIO1 when $N_H = 1$ and $S_{1f} = 1$ for every f. In this sense, MAIO1 households are recovered exactly as the limit case in which restriction A1 of MAIO1 holds.

2.2 Non-financial firms

Firms are institutional units that own production units. They serve as a fiscal and accounting layer: dividends, taxes, and—in MAIO2—loan repayments are settled at the firm level, while production decisions are delegated to the KAUs that the firm owns.

State. Each firm f is characterised by its own money stock $M_{f,t}$, a collection of owned KAUs indexed by $k \in \mathcal{K}_f$, the firm-to-KAU quota matrix $\Omega_{fk} \in [0,1]$ with $\sum_f \Omega_{fk} = 1$ for every k, and—new in MAIO2—a list of outstanding loans to the bank, each described by a residual principal, an interest rate, and a maturity. Firm f's total money is

$$M_{f,t} = \sum_{k \in \mathcal{K}_f} \Omega_{fk} \, M_{k,t}, \tag{6}$$

which is identical to MAIO1.

Decision rules. The firm performs three actions per period: (i) at the beginning, it distributes dividends to households out of the previous period's profits, allocated according to ownership shares (equation (1)); (ii) it allocates current liquidity across the KAUs it owns, financing their input procurement, wage bill, and capital expenditure; (iii) at the end of the period, it consolidates the accounts of its KAUs into firm-level revenues, costs, profits and—new in MAIO2—it services the outstanding loans according to the amortisation schedule fixed at issuance.

The dividend distribution rule, the within-firm liquidity allocation, and the consolidation step are unchanged with respect to MAIO1. The credit relationship is described in Section 2.4; the tax obligation, in Section 2.5.

Interactions. Firms interact with households (dividend distribution and ownership), with their KAUs (intra-firm liquidity allocation and consolidation), with the bank (loans and repayments) and with the government (tax payments). Firms do not interact directly with the goods market: production and sales are operated by their KAUs.

Remark. The firm layer of MAIO2 is a strict extension of that of MAIO1: the only structural addition is the loan ledger and the corresponding amortisation flows. When the credit market is inactive (an explicit C-condition, introduced in Section 4), MAIO2 firms reduce to MAIO1 firms.

2.3 Local KAUs

Local kind-of-activity units (KAUs) are the productive agents of the economy. Each KAU produces exactly one product type and is owned by a single firm. The structural extension of MAIO2 is concentrated here: production now requires fixed capital and labour as complementary factors alongside intermediate inputs; investment in capital follows a Sraffian-supermultiplier accelerator; and prices become endogenous, emerging as a markup over a unit cost that includes the wage bill. The other behavioural rules (demand expectation, inventory targets, sale ordering) are inherited from MAIO1 with at most notational adjustments.

State. At time t, each local KAU k producing in sector i is characterised by the following stocks and prices:

- the input inventories $I_{jk,t}$ of intermediate consumption goods, for each input type $j = 1, \ldots, N_A$;
- the capital stocks $K_{jk,t}$ of fixed capital embodied in commodities of type $j = 1, \ldots, N_A$;
- the workforce $N_{k,t}$, the number of employed workers;
- the money stock $M_{k,t}$, allocated by the owning firm;
- the posted price $p_{k,t}$ at which the KAU sells its own product, and the vector of input prices $(p_{jk,t})_j$ paid on the previous purchase round;
- the expected demand $q^{De}_{k,t}$, formed at the beginning of the period.

Each KAU is characterised by the following constant parameters, all sector-specific (we drop the firm subscript on parameters when one KAU operates in each sector): the technical coefficients C_{ji} (intermediate input requirements), the capital intensities γ_{ji} (fixed-capital requirements), the labour intensity C_{wi}, the depreciation rates δ_{ji} of each capital good, the target capital utilisation rates $\hat{c}_{ji}$, the inventory-adjustment speed v_i, the capital-adjustment speed $\rho_i \in [0,1]$, and the markup-adjustment sensitivity $s_i \geq 0$.

Decision rules. The KAU performs, in order, the operations described below.

Demand expectation. The KAU forms a forecast of next-period demand on the basis of past realised demands, exactly as in MAIO1 (equation (2) therein):

$$q^{De}_{k,t} \;=\; \sum_{t'=1}^{\tau_k} \beta_{k,t'}\, q^{D}_{k,t-t'}, \tag{7}$$

with weights $(\beta_{k,t'})$ summing to one over the look-back window τ_k.

Production capacity and plan. Production requires three complementary inputs: intermediate goods, capital, and labour. Following the bottleneck principle of Leontief technology, the production capacity $q^{\max}_{k,t}$ of KAU k in sector i is

$$q^{\max}_{k,t} \;=\; \min\!\left(\min_{j=1,\dots,N_A} \left\{ \frac{K_{jk,t}}{\gamma_{ji}} \right\},\; \frac{N_{k,t}}{C_{wi}},\; \min_{j=1,\dots,N_A} \left\{ \frac{I_{jk,t}}{C_{ji}} \right\} \right), \tag{8}$$

where the three arguments of the outer min correspond to the capital, labour, and intermediate-input bottlenecks. The actual production $q_{k,t}$ is the minimum of the production capacity and the production plan $\hat{q}_{k,t}$:

$$q_{k,t} \;=\; \min\!\left(q^{\max}_{k,t},\, \hat{q}_{k,t}\right). \tag{9}$$

The production plan $\hat{q}_{k,t}$ is set, as in MAIO1, to meet expected demand and to restore intermediate-input inventories to their medium-term target. Inventory targets and the resulting plan equations are inherited from MAIO1 (equations (3)–(6) therein) and are not restated here.

Labour demand. Given the production plan, the labour requirement of KAU k is $\hat{N}_{k,t} = \lceil C_{wi}\, \hat{q}_{k,t} \rceil$. The labour demand is the (non-negative) gap with respect to the current workforce:

$$\Delta N^{d}_{k,t} \;=\; \max\!\Big(0,\; \hat{N}_{k,t} - N_{k,t-1}\Big). \tag{10}$$

If the gap is negative, the KAU lays off $|\hat{N}_{k,t} - N_{k,t-1}|$ workers, selected at random. The decentralised matching of vacancies and workers is described in Section 3.2.

Capital target and investment demand. The desired endowment $\hat{K}_{jk,t}$ of capital good j for KAU k is determined by the production plan, the capital intensity, and the target utilisation rate:

$$\hat{K}_{jk,t} \;=\; \frac{\gamma_{ji}\, \hat{q}_{k,t}}{\hat{c}_{ji}}. \tag{11}$$

Both the capital intensity γ_{ji} and the target utilisation rate $\hat{c}_{ji}$ are defined on the gross output $\hat{q}_{k,t}$, which is the total physical production of KAU k inclusive of the share used internally as intermediate input.[1] The investment demand $\Delta K^{d}_{jk,t}$ is the sum of two components: a replacement term covering depreciation, and an adjustment term covering the gap with respect to the target,

$$\Delta K^{d}_{jk,t} \;=\; \delta_{ji}\, K_{jk,t-1} \;+\; \rho_i \left(\hat{K}_{jk,t} - K_{jk,t-1}\right), \tag{12}$$

with $\rho_i \in [0,1]$ governing the speed of adjustment toward the target. This formulation generalises the supermultiplier-accelerator specification used in [8]: replacement is unconditional, adjustment

[1] The relation between gross output and net (externally directed) demand at the steady state is $\hat{q}_{i*} = q^{D}_{i*}/(1 - C_{ii})$ and is derived in Section 5.1; under restriction B3 of the benchmark, this relation extends to every period.

is partial. When $\rho_i = 1$ and at zero depreciation, (12) reduces to the bare accelerator $\Delta K^d_{jk,t} = \hat{K}_{jk,t} - K_{jk,t-1}$.

Intermediate-input demand. The planned production also determines the demand for intermediate inputs, through the technical coefficients, augmented by an inventory-restocking term:

$$q^{ID}_{jk,t} \;=\; C_{ji}\,\hat{q}_{k,t} \;+\; v_i\big(I^T_{jk,t} - I_{jk,t-1}\big), \quad j \neq i, \tag{13}$$

where $I^T_{jk,t}$ is the medium-term inventory target inherited from MAIO1 (equations (3)–(4) therein). The own product is excluded from this expression: own inventories are managed implicitly through the production plan.

Equation (13) differs from its MAIO1 counterpart [1, eq. (9)] in that it scales the input demand on the *planned* production $\hat{q}_{k,t}$ rather than on the realised production $q_{k,t}$. This is a deliberate consequence of the credit-request mechanism introduced in this term of the sequence: the firm secures liquidity from the bank *before* production takes place (Section 3.3, phase 4 of Table 1), and the cash advanced to each KAU is dimensioned on the production plan. Tying the input demand to $\hat{q}_{k,t}$ ensures that the cash advanced for input purchases is fully spent on input purchases, closing the accounting cycle within the period. The same logic applies to the wage bill, which is also computed on the production plan (equation (30)). At the steady state $\hat{q}_{i*} = q_{i*}$ and the specification (13) coincides exactly with the MAIO1 formulation; off the steady state, when realised production falls short of the plan (input shortage or labour rationing), the specification produces an automatic accumulation of intermediate inventories above their target. This accumulation is a deliberate feature of the buffer-stock interpretation of the model and is consistent with the treatment of own-product inventories in MAIO1 (equation (5) therein).

Pricing. The unit cost of production of KAU k at time t includes both the intermediate inputs and the wage bill,

$$uc_{k,t} \;=\; \sum_j C_{ji}\,p_{jk,t-1} \;+\; C_{wi}\,w_{k,t}, \tag{14}$$

where $w_{k,t}$ is the wage paid by KAU k in period t. Equation (14) extends equation (11) of MAIO1 by adding the labour cost. The price is set as a markup over the unit cost,

$$p_{k,t} \;=\; (1 + \mu_{k,t})\,uc_{k,t}, \qquad \mu_{k,t} \geq 0, \tag{15}$$

where the markup $\mu_{k,t}$ adjusts in response to the deviation of realised from expected demand,

$$\mu_{k,t} \;=\; \mu_{k,t-1} \;+\; s_i \cdot f\left(\frac{q^D_{k,t} - q^{De}_{k,t}}{q^D_{k,t}}\right), \tag{16}$$

subject to $\mu_{k,t} \geq 0$, where $f(\cdot)$ is sign-preserving and $s_i \geq 0$ is the markup sensitivity. The B1 restriction of MAIO1 is recovered by setting $s_i = 0$, in which case the markup—and hence the price—remains constant.

Interactions. KAUs interact with five other agent types: with their owning firm (allocation of liquidity and consolidation of accounts, see Section 2.2); with other KAUs in the goods market, both as buyers (intermediate inputs and capital goods) and as sellers, following the decentralised protocol of Section 3.1; with households in the goods market, as sellers; with households in the labour market, as employers, through the decentralised protocol of Section 3.2; and with the bank, indirectly via the firm, when the latter requires loans to finance investment plans (Section 3.3).

Remark. Equations (7)–(13) reduce to the corresponding equations of MAIO1 when the labour and capital bottlenecks are not binding—in particular, when capital and labour are abundant relative to the intermediate-input bottleneck. In that limit, equation (8) collapses to the third argument of the outer minimum, and the model behaves exactly as the first term of the sequence with respect to the goods market. Equation (15) reduces to fixed prices under $s_i = 0$ (restriction B1 of MAIO1).

2.4 Banks

The economy hosts a set $\mathcal{B}$ of commercial banks, with $|\mathcal{B}| = N_B \geq 1$, introduced for the first time in this term of the MAIO sequence. Each bank holds the deposits of a subset of firms and households, grants loans to its client firms, and channels its interest income back to its owning households as dividends. We specify here the role of a generic bank $b \in \mathcal{B}$ in the multi-bank framework; the benchmark of Section 5 specialises the framework to $N_B = 1$ via restriction A4 (Section 4.2).

The bank–firm and bank–household relations are organised through two distinct concepts. Each firm f and each household h has a *primary bank* $b(f), b(h) \in \mathcal{B}$, with which it maintains its deposit account and through which all money flows of period t are settled; primary banks are assigned at initialisation, uniformly at random across $\mathcal{B}$, and are not re-evaluated within the simulation. A firm holds its deposits and addresses its credit requests to the same primary bank $b(f)$: deposit and lending relations are not split across banks. Separately, each household carries a vector of *bank ownership shares* $(S_{hb})_{b\in\mathcal{B}}$ over the entire set of banks, with $\sum_h S_{hb} = 1$ for every b, governing the redistribution of bank profits. The two concepts are in general independent: a household with primary bank $b(h) = b_0$ may hold ownership shares in any subset of banks.

State. At time t, bank b is described by:

- a vector of deposit balances $(D_{a,t})_{a\in\mathcal{C}_b}$ held by its clients $\mathcal{C}_b = \{a : b(a) = b\}$, including both firms and households;
- a portfolio of outstanding loans $\mathcal{L}_t^b = \{\ell_{n,t}\}_{n:b(n)=b}$, each loan $\ell_{n,t}$ characterised by a granted principal L_n, a residual principal $L_{n,t}^R$, an interest rate r_b, the borrowing firm $f(n)$ (with $b(f(n)) = b$), and a residual maturity $T_{n,t}^R$;
- a stock of reserves $R_{b,t}$, accumulating loan repayments and depleting on dividend distribution;
- the cumulative interest income $\Pi_{b,t}^B$ of the period.

Each bank $b \in \mathcal{B}$ is parametrised by its own interest rate r_b, loan maturity T_b, and binary lending threshold $\theta_b \in \{0, 1\}$, stored as separate agent attributes (`Banks_var` in the code base). When $\theta_b = 1$ bank b grants whatever its client firms request; when $\theta_b = 0$ no credit is granted at all. The current implementation initialises every bank with identical values for these three parameters, replicating the entries of a single-valued parameter vector; the uniformity assumption is formalised in Section 4.2 as restriction A5 of the benchmark. The analytical framework allows structurally for per-bank heterogeneity, but its exercise would require extending the parameter vectors and is not undertaken in the present term of the sequence. Partial granting (proper credit rationing) is also not implemented; we comment on this in the remark at the end of the subsection.

Decision rules. Each bank performs three actions per period.

Loan granting. Each client firm $f \in \mathcal{C}_b$ submits a credit request $L_{f,t}^d$ that aggregates the residual liquidity needs of its KAUs (see Section 2.2). On receiving the request, the firm's primary bank $b = b(f)$ applies its granting rule:

$$L_{f,t}^{\text{granted}} = \begin{cases} L_{f,t}^d & \text{if } \theta_{b(f)} = 1, \\ 0 & \text{if } \theta_{b(f)} = 0. \end{cases} \tag{17}$$

When granted, the loan is added to the bank's portfolio with residual principal $L_{n,t}^R = L_{f,t}^d$ and residual maturity $T_{n,t}^R = T_{b(f)}$. The granted amount is credited to the firm's deposit at its primary bank, $D_{f,t} \to D_{f,t} + L_{f,t}^{\text{granted}}$.

Repayment. For each outstanding loan ℓ_n with residual maturity $T^R_{n,t} > 0$, the firm pays a fixed instalment to the issuing bank $b(n)$ according to the standard French amortisation schedule, computed at the bank's interest rate $r = r_{b(n)}$:

$$P_{n,t} \;=\; \frac{r}{1-(1+r)^{-T^R_{n,t}}}\, L^R_{n,t-1}, \qquad I_{n,t} = r\, L^R_{n,t-1}, \qquad C_{n,t} = P_{n,t} - I_{n,t}, \tag{18}$$

where $P_{n,t}$ is the total instalment, $I_{n,t}$ the interest component, and $C_{n,t}$ the principal repayment. The case $r = 0$ is treated separately as $P_{n,t} = L^R_{n,t-1}/T^R_{n,t}$ (linear repayment of principal). The residual principal is updated as $L^R_{n,t} = (1+r)\, L^R_{n,t-1} - P_{n,t}$ and the residual maturity is decremented; loans with $T^R_{n,t} = 0$ are removed from the portfolio of bank $b(n)$.

Dividend distribution. At the end of the period, each bank distributes its accumulated interest income $\Pi^B_{b,t} = \sum_{n:b(n)=b} I_{n,t}$ to the household population according to its ownership-share vector:

$$\Pi^B_{h,t} \;=\; \sum_{b\in\mathcal{B}} S_{hb}\, \Pi^B_{b,t}. \tag{19}$$

The dividend received by household h is credited to its deposit account at its primary bank $b(h)$, regardless of where the underlying interest income was earned. Under the uniform specialisation $S_{hb} = 1/N_H$ for every (h, b), which is the specification adopted in the current implementation, equation (19) reduces to the equal-share rule $\Pi^B_{h,t} = (1/N_H)\,\sum_b \Pi^B_{b,t}$.

Interactions. Banks interact with non-financial firms through the deposit and loan channels of equations (17)–(18), and with households through the deposit channel (for income and consumption settlement) and the dividend channel (19). Inter-bank transactions are absent in the current implementation; interactions with the government and the rest of the world go through firms and households, never directly.

Remark. Three features of the present specification deserve emphasis. First, the lending rule (17) is binary: full granting or full refusal. The credit market admits no quantity-rationing mechanism, which places it in a special position relative to the goods market and the labour market, where rationing emerges naturally when supply falls short of demand. Condition C3 of Section 4.4 distinguishes the credit-active regime $\theta_b = 1$ for all b from the credit-inactive regime $\theta_b = 0$; the latter recovers the firm structure of MAIO1. Second, the matching of firms and households to primary banks is performed only at initialisation and is never re-evaluated; preferred-bank protocols with threshold-based switching, analogous to the goods-market protocol of Section 3.1, are not implemented in this term and are natural targets for subsequent terms of the sequence. Third, the bank ownership shares S_{hb} are calibrated to the uniform specialisation $S_{hb} = 1/N_H$ in the current implementation, which is the simplest specification compatible with the aggregation theory of Section 6; a non-uniform specification, e.g., proportional to deposits or to the wealth distribution, would introduce a non-trivial coupling between the wealth distribution and the bank-dividend channel that we discuss in Section 6.2.

2.5 Government

The government is a single institutional agent that levies taxes on production and on labour income, distributes transfers and unemployment benefits to households, and consumes sectoral output. Together with the rest of the world (Section 2.6), it closes the macroeconomic circuit of MAIO2.

State. At time t, the government is described by:

- the liquidity stock L^G_t, accumulating tax revenues and depleting on consumption and transfers;

- the consumption budgets $(B^G_{i,t})_{i=1,\dots,N_A}$, allocated across sectors at the beginning of each period;
- the period flows of transfers paid, unemployment benefits paid, and taxes received.

The government is parametrised by a vector of consumption shares $(\alpha^G_i)_{i=1,\dots,N_A}$ summing to one, a vector of VAT rates $(\tau^V_i)_{i=1,\dots,N_A}$, a corporate tax rate τ^C, the transfer parameter ϕ^T (a fraction of the average wage), the unemployment-benefit parameter ϕ^U, and an import quota for government purchases $q^G \in [0,1]$.

Decision rules. The government performs four actions per period.

Tax collection. Taxes are levied on KAU revenues and net earnings. Each KAU k producing in sector i pays

$$T^V_{k,t} \;=\; \frac{\tau^V_i}{1+\tau^V_i}\, R_{k,t}, \qquad T^C_{k,t} \;=\; \tau^C \left(R_{k,t} - T^V_{k,t} - \mathrm{Cost}_{k,t}\right)^+, \tag{20}$$

where $R_{k,t}$ is the period revenue and $\mathrm{Cost}_{k,t}$ the period cost (intermediate inputs, wages, depreciation), and $(x)^+ = \max(0,x)$. Taxes are paid out of the firm's deposit at the bank and credited to the government's liquidity:

$$L^G_t \;=\; L^G_{t-1} \;+\; \sum_k \bigl(T^V_{k,t} + T^C_{k,t}\bigr). \tag{21}$$

Transfers and unemployment benefits. The government distributes transfers to all households and unemployment benefits to the unemployed subset, according to the average wage $\bar{w}_t$ of the economy:

$$T_{h,t} \;=\; \phi^T\, \bar{w}_t, \qquad U_{h,t} \;=\; \phi^U\, \bar{w}_t \cdot \mathbb{1}[h \text{ unemployed at } t]\,, \tag{22}$$

subject to the liquidity constraint: if L^G_t is insufficient to cover the planned aggregate transfer, the per-household amount is rescaled proportionally so that the total never exceeds available liquidity. This soft constraint is read directly from the implementation; it does *not* amount to a buffer-stock rule for government spending.

Budget allocation and consumption. The government allocates its remaining liquidity across sectoral consumption budgets according to fixed shares,

$$B^G_{i,t} \;=\; \alpha^G_i\, L^G_t, \qquad \sum_i \alpha^G_i = 1. \tag{23}$$

A fraction q^G of each $B^G_{i,t}$ is spent domestically (on the local KAUs of sector i), the residual fraction $(1-q^G)$ on foreign imports purchased from the rest of the world.

Interactions. The government interacts with KAUs as a tax collector (equation (20)) and as a buyer in the goods market (domestic share of equation (23)); with households as a transfer agent (equation (22)); and with the rest of the world as an importer (foreign share of equation (23)).

Remark. The government spending rule of equation (23) is "spend all available liquidity": the government carries no precautionary savings and runs no fiscal stabilisation rule. In particular, no buffer-stock target analogous to λ^T_h is imposed on the government. Together with the soft transfer constraint, this means that the government's role is essentially that of an automatic redistributor between firms (taxes), households (transfers and benefits), and KAUs (public consumption). A C-condition that switches off the government recovers the household–KAU economy of MAIO1, augmented by capital, labour, and the rest of the world.

2.6 Rest of the World

The rest of the world (RoW) is a single aggregate agent that absorbs domestic exports and supplies foreign imports. It is the second new institutional agent of MAIO2 and, together with the government, internalises the open-economy dimension that MAIO1 leaves implicit.

State. At time t, the RoW is described by:

- the foreign GDP Y_t^R, which is exogenous up to a constant growth rate g^R that we set to zero in the benchmark configuration;
- the consumption budgets $(B_{i,t}^R)_{i=1,\dots,N_A}$ allocated to domestic exports;
- the foreign liquidity L_t^R, accumulating import revenues and depleting on export purchases (an accounting device that closes the trade balance);
- a flow vector of period sales $\{s_{i,t}^R\}$, used as a supplier-of-last-resort to domestic buyers.

The RoW is parametrised by a constant consumption-to-GDP ratio c^R, a vector of expenditure shares $(\alpha_i^R)_{i=1,\dots,N_A}$ summing to one, a fixed vector of import prices $(p_i^R)_{i=1,\dots,N_A}$ at which the RoW supplies foreign goods, and a matrix of supplier weights (ω_{ki}^R) that distribute foreign demand for sector i across domestic KAUs producing in that sector.

Decision rules. The RoW performs two actions per period.

Foreign demand on domestic exports. The aggregate consumption budget of the RoW is

$$B_t^R \;=\; c^R\, Y_t^R, \qquad B_{i,t}^R \;=\; \alpha_i^R\, B_t^R. \tag{24}$$

Within sector i, the budget is allocated to domestic KAUs according to the (calibrated) supplier weights ω_{ki}^R:

$$E_{ki,t} \;=\; \omega_{ki}^R\, B_{i,t}^R, \qquad \sum_{k\in\mathcal{K}_i} \omega_{ki}^R = 1, \tag{25}$$

where $\mathcal{K}_i$ is the set of KAUs operating in sector i. Each $E_{ki,t}$ is converted into a physical export demand using the seller's posted price $p_{k,t}$. Unmet demand is not redirected to alternative suppliers within the period: the RoW makes a single purchasing pass per commodity.

Foreign supply at fixed prices. On the supply side, the RoW provides imports to domestic buyers at the constant prices p_i^R. The RoW supply is unbounded ($\bar{s}_i^R = +\infty$ for every i): when domestic KAUs cannot fulfil a buyer's demand for product i, the residual demand is met by the RoW. In this sense, the RoW acts as a supplier of last resort, and its sales $s_{i,t}^R$ are determined endogenously by the residual demand of domestic agents.

Interactions. The RoW interacts with KAUs both as a buyer of domestic exports (equations (24)–(25)) and as a seller of imports at fixed prices, with all domestic agents (households, KAUs, government) that are rationed in the goods market.

Remark. The RoW liquidity L_t^R tracks the cumulative trade balance: it increases when domestic agents buy imports (RoW receives payment) and decreases when the RoW buys exports (RoW spends). In the benchmark of Section 5, where $g^R = 0$ and trade flows are constant, L_t^R converges to a stationary level that mirrors the steady-state trade balance.

3 Decentralised markets

This section describes the three decentralised markets through which the agents of Section 2 interact: the goods market (Section 3.1), where buyers and sellers of intermediate, capital, and final goods meet across the N_A commodities; the labour market (Section 3.2), where KAUs hire and households supply labour services; and the credit market (Section 3.3), where firms aggregate their KAUs' funding needs and the bank applies its lending rule. The three markets share the structural features of decentralised matching and potential rationing, but differ in the matching protocol: a preferred-supplier cascade with switching for the goods market, a random matching for the labour market, and a single-bank binary granting for the credit market. The period timing that orders these markets within a single period of the simulation is the object of Section 4.

3.1 Goods market

The goods market of MAIO2 generalises the decentralised protocol of MAIO1 [1, §1.1] along two dimensions. First, the set of buyers is enlarged to include the Government (Section 2.5) and the rest of the world (Section 2.6); the set of sellers retains the domestic KAUs and adds the RoW as a foreign supplier. Second, the demand of each buyer for each commodity is split between a domestic component and a foreign component according to an exogenous import quota. With these two changes, the four-step structure of MAIO1 (preferred supplier → cascading rationing → switching → endogenous bilateral network) is preserved as a framework, although the benchmark configuration of Section 5 specialises it in a way that we make explicit at the end of the subsection.

Sellers, buyers, and the bipartite graph. For each commodity $i \in \{1, \dots, N_A\}$, the set of *potential sellers* consists of the domestic KAUs producing i, denoted $\mathcal{K}_i$, together with the rest of the world acting as a foreign supplier. The set of *buyers* of commodity i consists of four classes:

- the domestic KAUs that use i as an intermediate input or as a capital good, indexed by $k \in \mathcal{K} \setminus \mathcal{K}_i$;
- the households $h = 1, \dots, N_H$ that consume i with positive expenditure share;
- the Government, which spends part of its budget on i;
- the rest of the world, which absorbs part of the domestic production of i as exports.

The interaction between buyers and sellers across the N_A commodities defines a time-varying bipartite graph, the *purchasing network*, distinct from the ownership network of Figure 1 of MAIO1. The four buyer classes follow different allocation rules, which we describe below.

Preferred-supplier rule and cascade (KAUs and households). Each KAU and each household maintains, for every commodity i that it purchases, a finite ordered list of known sellers $\mathcal{S}_b^{(i)} \subseteq \mathcal{K}_i \cup \{\text{RoW}\}$, constructed at initialisation from a random subset of size $\lceil \phi_b \, |\mathcal{K}_i| \rceil$ of the domestic producers (parameter $\phi_b \in (0, 1]$, the buyer's *field of view*), extended by the RoW as a fallback supplier. The first element of $\mathcal{S}_b^{(i)}$ that is not the RoW is the *preferred domestic supplier* of b for commodity i, denoted $s_b^*(i) \in \mathcal{K}_i$. Between purchase rounds the list is periodically re-sorted by current posted prices (function `update_sellers_list` in the code base).

The market period proceeds in R purchase rounds. In each round, every buyer with residual demand contacts its preferred domestic supplier first: if the latter's available supply is sufficient, the transaction clears in full; otherwise the buyer is rationed and the unsatisfied portion is redirected to the next supplier in $\mathcal{S}_b^{(i)}$ with positive supply, until either the demand is met or the list (RoW included) is exhausted. The RoW, by construction, has unbounded supply (Section 2.6), so the cascade always terminates within a single round.

Domestic–foreign quota and the role of the RoW as last resort. A new feature of MAIO2 is that each non-Government buyer splits its demand for commodity i between domestic and foreign supply according to an exogenous *import quota*. Let $q_b^D(i) \in [0, 1]$ denote the domestic quota of buyer b for commodity i, with foreign quota $q_b^R(i) = 1 - q_b^D(i)$. For KAUs the quota acts on physical quantities (parameter `qD_intermediate`); for households it acts on the nominal consumption budget (parameter `qD_final`). Concretely, given a total nominal demand $D_b(i)$ for commodity i, the buyer first directs $q_b^D(i)\, D_b(i)$ to the preferred domestic supplier; the foreign quota $q_b^R(i)\, D_b(i)$ is allocated to the RoW at the posted import price p_i^R. Any rationing on the domestic component is added on top of the planned foreign quota: the residual demand left by the cascade on $\mathcal{S}_b^{(i)}$ is always absorbed by the RoW. The RoW therefore plays a double role: it is the destination of the buyer's foreign quota by design, and the supplier of last resort whenever domestic supply falls short of the planned domestic quota.

Government allocation rule. The Government does not maintain a preferred-supplier list. Given its sectoral budget $B_{i,t}^G$ (Section 2.5) and its sectoral domestic quota $q_G^D(i)$, the domestic share $q_G^D(i)\, B_{i,t}^G$ is allocated *uniformly* across the domestic producers of i with positive supply, $\mathcal{K}_i^+(t) = \{k \in \mathcal{K}_i : \text{supply}_{k,t} > 0\}$:

$$B_{ki,t}^G \;=\; \frac{q_G^D(i)\, B_{i,t}^G}{|\mathcal{K}_i^+(t)|}, \qquad k \in \mathcal{K}_i^+(t), \tag{26}$$

the foreign share $q_G^R(i)\, B_{i,t}^G$ together with any rationing shortfall accruing during the domestic round is purchased from the RoW. The Government's purchasing relation is therefore a complete bipartite edge between the public buyer and the set of active producers in each sector, with uniform weights.

Rest-of-the-world allocation rule. The RoW as a buyer (Section 2.6) allocates its sectoral import budget $B_{i,t}^R$ across the domestic producers of i according to the calibrated weights $(\omega_{ki}^R)_{k\in\mathcal{K}_i}$ introduced in equation (25),

$$E_{ki,t} \;=\; \omega_{ki}^R\, B_{i,t}^R, \qquad \sum_{k\in\mathcal{K}_i} \omega_{ki}^R = 1, \tag{27}$$

with a single purchasing pass per commodity (no cascade): if a domestic producer cannot fulfil its assigned share, the residual is forgone and not redirected to other producers within the period.

Switching threshold and the endogenous network. As in MAIO1, the buyer–seller network can evolve through two switching mechanisms acting on the preferred-supplier assignment of KAUs and households. First, when the rationing experienced at the preferred supplier exceeds a threshold $\bar{\rho}_b \in [0,1]$, the buyer replaces $s_b^*(i)$ with the next element of $\mathcal{S}_b^{(i)}$ having positive supply. Second, with probability $\omega_b \in [0,1]$, the buyer reorders $\mathcal{S}_b^{(i)}$ by current posted price and adopts the cheapest known seller as new preferred supplier. The parameters $\bar{\rho}_b, \omega_b$ are individual attributes of each buyer and were introduced in MAIO1 as the *switching threshold* and the *opportunism degree*, respectively. The list itself rotates slowly through a *memory-loss* parameter $\mu_b \in [0,1]$ that, at each market opening, replaces a fraction μ_b of the tail of $\mathcal{S}_b^{(i)}$ with newly sampled domestic candidates. The combination of cascading rationing, switching, and memory loss generates an endogenously evolving purchasing network whose dynamics are responsible for the path dependence documented in [1, §2.4–2.5].

Period timing of the market. Within a period, all buyers share the same market window. Buyers contact sellers in random order, repeated over R rounds; in each round, every buyer with residual demand executes one purchase attempt on a given commodity (function `buy` in the implementation). The commodity ordering is shared across all buyers and is documented in the period timing of Section 4; rounds across commodities are sequential. At the end of the market window, sellers update their inventories and revenues; buyers update their stocks of inputs, capital goods, and money.

Remark (Specialisation under the benchmark). Under restriction A2 of Assumption 1 (one KAU per sector), each sector contains a unique domestic seller and the preferred-supplier rule degenerates: the cascade on $\mathcal{S}_b^{(i)}$ collapses to a single domestic seller followed by the RoW, and the switching threshold becomes structurally inactive (there is no alternative domestic supplier on which to switch). The purchasing network reduces to a star graph centred on the unique producer of each sector. The endogenous network dynamics of MAIO1 re-emerge in subsequent terms of the sequence, where A2 is relaxed and each sector hosts multiple competing KAUs.

3.2 Labour market

The labour market of MAIO2 is the first of the two markets that have no analogue in MAIO1. Production now requires labour as a complementary factor (Section 2.3); the labour market is the mechanism that matches the workforce demanded by the productive agents—the local KAUs—with the workforce supplied by households. Relative to the goods market of Section 3.1, the labour market of MAIO2 adopts a simpler matching protocol: it is decentralised and rationable, but it features no preferred-counterparty mechanism, no switching threshold, and no wage bargaining. Wages are posted by each KAU and are fixed throughout the simulations of this paper; labour–demand mismatches are resolved through unemployment on the household side and through unfilled vacancies on the KAU side.

Buyers, sellers, and the bipartite matching. On the demand side stand the domestic KAUs $k \in \mathcal{K}$, each characterised by a wage offer $w_k > 0$, a labour intensity $C_{wk} > 0$ (Section 2.3), and an employee roster $\mathcal{N}_{k,t}$ of size $N_{k,t} = |\mathcal{N}_{k,t}|$. On the supply side stand the households $h = 1, \dots, N_H$, each characterised by an employment status $e_{h,t} \in \{0, 1\}$ (with $e_{h,t} = 1$ if employed at the beginning of period t) and, if employed, by the identity of the employing KAU $k(h,t) \in \mathcal{K}$. Government and the rest of the world do not participate in the labour market in the present specification.

The matching is realised through a one-shot, decentralised meeting protocol described below: in each period, KAUs with unfilled vacancies post their openings; unemployed households contact a randomly drawn employer with an open vacancy; matches are formed sequentially until either all vacancies are filled or all unemployed households have attempted. There is no second round.

Labour demand and the firing rule. At the beginning of period t, each KAU k producing in sector i forms its production plan $\hat{q}_{k,t}$ (Section 2.3, equation (9)). The associated labour requirement is $\hat{N}_{k,t} = \lceil C_{wi}\, \hat{q}_{k,t} \rceil$; the gap with respect to the current workforce determines whether the KAU posts a vacancy or fires workers,

$$\Delta N_{k,t} = \hat{N}_{k,t} - N_{k,t-1}, \tag{28}$$

with the following adjustment rule:

$$\begin{cases} \text{If } \Delta N_{k,t} < 0: & \text{the KAU dismisses } |\Delta N_{k,t}| \text{ workers, drawn uniformly at random from } \mathcal{N}_{k,t-1}, \\ & \text{and posts no vacancy } (v_{k,t} = 0). \\ \text{If } \Delta N_{k,t} \geq 0: & \text{the KAU posts a vacancy with } v_{k,t} = 1 \text{ and labour-demand} \\ & \text{quantity } \Delta N^d_{k,t} = \Delta N_{k,t}, \text{ to be filled in the matching round.} \end{cases} \tag{29}$$

Dismissed workers update their status to $e_{h,t} = 0$ and become available for the matching round. The labour-demand quantity $\Delta N^d_{k,t}$ records the number of new hires the KAU is willing to absorb in period t.

Labour supply and the matching protocol. Let $\mathcal{V}_t = \{k \in \mathcal{K} : v_{k,t} = 1\}$ denote the set of KAUs with an open vacancy at time t, and $\mathcal{U}_t = \{h : e_{h,t} = 0\}$ the set of unemployed households at the same time (including those just dismissed by the firing rule). The matching protocol is the following:

(1) Each unemployed household $h \in \mathcal{U}_t$, in random order, observes $\mathcal{V}_t$ and selects an employer $k(h,t)$ uniformly at random from the KAUs still posting an open vacancy.

(2) The selected KAU adds h to its employee roster: $\mathcal{N}_{k(h,t),t} \to \mathcal{N}_{k(h,t),t-1} \cup \{h\}$; the residual labour-demand quantity is decremented by one, $\Delta N^d_{k(h,t),t} \to \Delta N^d_{k(h,t),t} - 1$; if the residual reaches zero, the vacancy is closed, $v_{k(h,t),t} \to 0$, and the KAU is removed from $\mathcal{V}_t$.

(3) The household updates its status to $e_{h,t} = 1$.

The protocol terminates when $\mathcal{V}_t$ is empty or when every household in $\mathcal{U}_t$ has attempted once. There is no reservation wage, no preferred-employer rule, and no second matching round within the period: an unemployed household that fails to match remains unemployed for the period, and a vacancy that remains open at the end of the matching round is carried over to the next period.

Wage formation. Each KAU posts an exogenous wage offer w_k, fixed at calibration and constant over the simulation. There is no within-period adjustment of the wage in response to vacancy rates or unemployment, and no bargaining at the time of matching. Wage heterogeneity across KAUs captures sectoral wage differentials when present in the calibration data, but in the benchmark of Section 5 the wage vector (w_k) is treated as a vector of constants. The wage offer enters the unit-cost equation (14) as the labour component of the KAU's marginal cost, and through it the posted price (equation (15)).

Wage payments. At the end of the production phase, each KAU disburses its wage bill. The total wage bill of KAU k in period t is set on the basis of the production plan, not of the realised employment:

$$W^{\text{tot}}_{k,t} \;=\; w_k\, C_{wi}\, \hat{q}_{k,t} \;=\; w_k\, \hat{N}_{k,t}. \tag{30}$$

The aggregate amount (30) is then distributed equally across the current employees of KAU k:

$$w^{\text{eff}}_{h,t} \;=\; \frac{W^{\text{tot}}_{k(h,t),t}}{N_{k(h,t),t}}, \qquad h \in \mathcal{N}_{k(h,t),t}. \tag{31}$$

The effective wage per worker coincides with the posted wage w_k when realised employment equals planned employment, $N_{k,t} = \hat{N}_{k,t}$; it exceeds w_k if the KAU is rationed in the labour market ($N_{k,t} < \hat{N}_{k,t}$), and falls below w_k if employment overshoots the plan. At the steady state of Section 5, the no-rationing condition ensures $N_{k,*} = \hat{N}_{k,*}$, so the effective wage coincides with the posted wage.

Production with the realised workforce. Once wages are paid, production proceeds. The Leontief technology of equation (8) treats labour as one of three complementary bottlenecks, with capacity $N_{k,t}/C_{wi}$ on the labour side. The actual production $q_{k,t}$ is then computed according to equation (9), with the caveat that a limited overtime margin $\bar{o} \in [0,1)$ allows the KAU to push realised production above the strict labour-bound capacity by a factor $(1+\bar{o})$:

$$q_{k,t} \;=\; \min\!\left(\hat{q}_{k,t},\; (1+\bar{o})\,\frac{N_{k,t}}{C_{wi}}\right). \tag{32}$$

Equation (32) subsumes the labour-capacity term of the outer minimum in (8); the parameter $\bar{o}$ (set to $\bar{o} = 0.3$ in the calibrations of this paper) captures the short-run elasticity of output to the realised workforce that goes beyond the nominal capacity, in the spirit of overtime work.

Period accounting. The wage bill of KAU k, equation (30), is debited from the cash account that the owning firm allocates to k, and is credited to the household account of each employee. Households add the received wage to their bank deposit, and the model proceeds to the goods market (Section 3.1).

Remark (Status of preferred-employer, switching, and bargaining). The labour market specified above departs from the four-step structure of the goods market in three respects: (i) there is no preferred employer and the buyer–seller graph carries no order beyond the binary matching of period t; (ii) there is no switching threshold, and an employed household does not re-evaluate its employer within or across periods; (iii) the wage offer is exogenous and is not adjusted in response to vacancy or unemployment rates. These simplifications are explicit modelling choices of the second term of the MAIO sequence and are intended to keep the analytical benchmark of Section 5 tractable. Under the no-rationing condition of Assumption 1, the labour market reaches its equilibrium configuration with full employment, $N_{k,*} = \hat{N}_{k,*}$, and the matching protocol clears the planned labour demand of each KAU without residual unemployment or unfilled vacancies. The labour market remains an active component of the period cycle—it is the mechanism that realises the labour allocation—but its outcomes at the benchmark are uniquely determined by the labour-demand vector $\hat{N}_*$. Endogenous wage formation, preferred-employer dynamics, and a richer search protocol are natural extensions left to subsequent terms of the sequence, in which the labour market becomes a source of frictions in its own right.

3.3 Credit market

The credit market closes the trio of markets specific to MAIO2 and differs structurally from the goods and labour markets in two respects. First, the credit network is centralised one-to-many: every firm has a unique primary bank $b(f) \in \mathcal{B}$ to which all its credit requests are addressed, and the bank–firm matching is fixed at initialisation (Section 2.4). Under restriction A4 (single bank, Section 4.2), the supply side reduces to a single agent and the preferred-counterparty rule of the goods market is structurally degenerate; we adopt A4 throughout this subsection and recover the multi-bank generalisation in subsequent terms of the sequence. Second, the granting decision is binary (Section 2.4, equation (17)): the bank either grants the firm's request in full or refuses it in full, and the within-period quantity-rationing dynamics of the goods market have no counterpart here. What remains genuinely interactive is the formation of the firm's credit request as an aggregation of its KAUs' funding needs, the within-firm allocation of the granted loan across the requesting KAUs, and the period accounting of repayments. We describe these mechanisms in turn.

Buyers, seller, and the funding gap. On the demand side stand the non-financial firms $f = 1, \ldots, N_F$ (Section 2.2); on the supply side stands the unique bank of the benchmark, denoted $b_0 \in \mathcal{B}$ under A4 (Section 2.4). Government, households, the rest of the world, and the KAUs do not participate as direct counterparties to the credit market: the firm is the unit of credit account.

At the beginning of period t, each KAU $k \in \mathcal{K}_f$ computes its own *money request*, which combines operating liquidity needs, the principal due on the firm's outstanding loans imputed to k, and the dividend payable out of k's net earnings, net of the cash already allocated to k by the firm. Formally,

$$M^d_{k,t} = \underbrace{\Lambda^{\mathrm{op}}_{k,t}}_{\text{operating needs}} + \underbrace{\sum_{n \in \mathcal{L}^f_t} w^L_{n,k}\, C_{n,t}}_{\text{principal repayments}} + \underbrace{\max\left(0,\, \pi^N_{k,t-1}\right)}_{\text{dividend payable}} - \underbrace{M^{\mathrm{cash}}_{k,t}}_{\text{cash on hand}}, \tag{33}$$

where $\Lambda^{\mathrm{op}}_{k,t}$ collects the cost of the planned inventory restocking of the own product, of the planned purchases of intermediate and capital goods, and of the planned wage bill; $\mathcal{L}^f_t$ is the set of outstanding loans of the firm owning k; $w^L_{n,k} \in [0,1]$ is the weight with which loan n was imputed to KAU k at issuance (introduced in (35) below), so that $\sum_{k\in\mathcal{K}_f} w^L_{n,k} = 1$; $C_{n,t}$ is the principal component of the period-t instalment on loan n (equation (18)); $\pi^N_{k,t-1}$ is the KAU's net earnings carried over from the previous period; and $M^{\text{cash}}_{k,t}$ is the cash balance allocated to k by the firm at the start of the period. The interest component $I_{n,t}$ of the instalment is netted directly against the dividend payable and is not financed through the credit market.

The firm aggregates the requests of its own KAUs into the period-t credit request:

$$L^d_{f,t} \;=\; \sum_{k\in\mathcal{K}_f} \max\Big(0,\; M^d_{k,t}\Big)\,. \tag{34}$$

The clipping at zero in (34) reflects the fact that the bank does not absorb excess cash: a KAU with a cash surplus does not contribute negatively to the firm's funding gap, and its surplus remains within the firm's deposit, available for intra-firm reallocation.

The bank's granting decision. The firm's primary bank $b(f)$ receives $L^d_{f,t}$ and applies the binary granting rule of Section 2.4, equation (17): under the active configuration of this paper, $\theta_{b(f)} = 1$ and the bank grants the request in full, $L^{\text{granted}}_{f,t} = L^d_{f,t}$; under the credit-inactive configuration $\theta_{b(f)} = 0$ and $L^{\text{granted}}_{f,t} = 0$. The granted amount is credited to the firm's deposit account at the same bank, $D_{f,t} \to D_{f,t} + L^{\text{granted}}_{f,t}$, and a new loan ℓ_n is added to the bank's portfolio with residual principal $L^R_{n,t} = L^{\text{granted}}_{f,t}$ and residual maturity $T^R_{n,t} = T_{b(f)}$.

Within-firm allocation of the granted loan. Each loan carries, alongside its principal and amortisation parameters, a within-firm weight vector $(w^L_{n,k})_{k\in\mathcal{K}_f}$ that records the share of the loan imputed to each KAU. The weights are determined at issuance, proportionally to the positive components of the KAUs' money requests:

$$w^L_{n,k} \;=\; \frac{\max(0,\, M^d_{k,t})}{\sum_{k'\in\mathcal{K}_f} \max(0,\, M^d_{k',t})}, \qquad k \in \mathcal{K}_f. \tag{35}$$

The weights are constant over the life of the loan and apply, in particular, to the principal repayments $C_{n,t}$ of subsequent periods through equation (33). The granted amount is then transferred from the firm's deposit into the cash account of each KAU according to its share: $M^{\text{cash}}_{k,t} \to M^{\text{cash}}_{k,t} + w^L_{n,k}\, L^{\text{granted}}_{f,t}$.

Repayments. At the financial-payments stage of period t, the firm services every outstanding loan whose residual maturity has not exceeded the bank's T. For each loan ℓ_n, the firm computes the instalment $P_{n,t}$ and its decomposition into capital and interest components $(C_{n,t}, I_{n,t})$ according to the French amortisation schedule of equation (18). The instalment is then debited from the KAUs' cash accounts according to the loan weights,

$$M^{\text{cash}}_{k,t} \;\to\; M^{\text{cash}}_{k,t} - w^L_{n,k}\, P_{n,t}, \qquad k \in \mathcal{K}_f, \tag{36}$$

and credited to the bank's reserves. The residual principal of the loan is updated as $L^R_{n,t} = (1+r)\, L^R_{n,t-1} - P_{n,t}$, and the residual maturity is decremented by one. The interest component $I_{n,t}$ accumulates in the bank's interest-income stock Π^B_t and is distributed to households at the end of the period (equation (19)).

Period accounting and the money flow. Putting the protocol together, the credit market generates the following circular flow over one period. (i) Each firm aggregates its KAUs' money requests through (34) and submits $L^d_{f,t}$ to the bank. (ii) The bank applies the granting rule (17) and credits $L^{\text{granted}}_{f,t}$ to the firm's deposit. (iii) The firm distributes the granted amount across its KAUs' cash accounts according to (35), storing the weights for future repayments. (iv) The KAUs use the cash to purchase intermediate inputs and capital goods (Section 3.1) and to pay wages (Section 3.2); the cash flowing out of the firm's deposit returns to it through the revenues of its KAUs. (v) At the end of the period, the firm services its outstanding loans through (36) and distributes net dividends to households (Section 2.2).

Remark (Structural absences and connection to the goods market). Three features distinguish the credit market from the goods market of Section 3.1 and require explicit mention. First, restriction A4 (single bank) makes the preferred-counterparty rule degenerate in the same sense as restriction A2 makes it degenerate in the goods market: there is no alternative supplier on which to switch. Second, the granting rule (17) is binary, so the cascading quantity-rationing protocol that operates on the goods market is structurally inactive on the credit side: the bank either accommodates the demand or shuts the credit line altogether, with no intermediate regime. Third, the bank performs no credit-worthiness assessment of the borrowing firm: the granting decision depends only on the parameter θ and not on the firm's balance sheet, current profitability, or outstanding debt. These three simplifications are internally consistent and place the credit market of MAIO2 in the post-Keynesian tradition of accommodative endogenous money, with a binary lending switch as the sole policy lever. Multi-bank competition (relaxing A4), partial granting, and a balance-sheet-driven granting rule are natural extensions left to subsequent terms of the sequence.

4 Period timing and behavioural restrictions

The previous two sections specify the agents and the decentralised markets that connect them. This section fixes two further pieces of the model that have so far been left implicit: the ordering of the events within a period (Section 4.1) and the restrictions under which the benchmark of Section 5 becomes analytically tractable. The latter are organised, in continuity with MAIO1 [1, §2.1], along three orthogonal dimensions: agent configurations (A-conditions, Section 4.2), behavioural rules (B-conditions, Section 4.3), and dynamic regimes (C-conditions, Section 4.4).

4.1 Sequence of events per period

Within each period t, the model performs the actions listed in Table 1 in the order indicated. The sequence is fully deterministic at the level of phases; the only randomness within a phase concerns the order in which agents of the same class act, when this is needed by the protocol (e.g. the random ordering of unemployed households in the labour-market matching of phase 5, and of buyers within each market round of phase 11). The table is read directly from the main loop of the implementation (`Model.step`) and mirrors, in its overall logic, Table 1 of MAIO1 [1].

Three features of Table 1 deserve a short comment. First, the credit market (phase 4) precedes the labour market (phase 5): the firm secures the liquidity to pay wages *before* the matching of workers takes place, so that the wage bill of phase 6 is funded with the loan disbursed at phase 4. Second, the wage payment of phase 6 is computed on the basis of the production plan, not of the realised matching outcome (eq. (30)): under-staffing therefore raises the effective wage per worker rather than scaling down the wage bill. Third, the goods market (phase 11) is preceded by the budget-formation phase 9: every buyer enters the market with a determined nominal expenditure, and rationing in the market translates into involuntary stock-building of liquid balances, not into a real-time revision of the budget.

4.2 Agent configurations: A-conditions

The agent-configuration conditions identify the structural multiplicity of the model. In continuity with MAIO1, we retain three A-conditions and introduce one new condition, A4, that concerns the financial sector:

- **A1.** *Homogeneous households.* Households are treated as a population of identical agents: every household h has the same consumption propensities $(c_h^y, c_h^w, \lambda_h^T)$, the same vector of commodity expenditure shares $\boldsymbol{\alpha}_h$, the same ownership structure $\boldsymbol{S}_h = (S_{hf})_{f=1,\ldots,N_F}$ over firms, and the same bank-ownership structure $\boldsymbol{S}_h^B = (S_{hb})_{b\in\mathcal{B}}$. Restriction A1 is relaxed in Section 5.6, where households are partitioned into a small number of homogeneous groups.
- **A2.** *One KAU per sector.* For every sector $i \in \{1,\ldots,N_A\}$, the set $\mathcal{K}_i$ of KAUs producing i is a singleton. We use the sectoral index i interchangeably as a KAU index throughout the steady-state analysis.
- **A3.** *One KAU per firm.* Each firm f owns exactly one KAU, $|\mathcal{K}_f| = 1$. Restriction A3 is *logically independent* of A2: in general, a firm may own multiple KAUs operating in distinct sectors, with the firm-to-KAU allocation governed by the quota matrix Ω_{fk} of Section 2.2. Under homogeneous households (A1), however, the composition $S_{hf}\,\Omega_{fk}$ that enters the dividend distribution (1) is uniform across households regardless of Ω_{fk}, so the firm-to-KAU partition is macroeconomically inert at the benchmark. Restriction A3 is therefore adopted as a notational convenience: combined with A2, it permits a one-to-one identification of firms, KAUs, and sectors, allowing us to drop the firm subscript on KAU-level variables, and reduces the within-firm credit allocation rule (35) to a tautology ($w_{n,k}^L = 1$ for the unique KAU of firm f).
- **A4.** *Single bank.* The set of banks $\mathcal{B}$ is a singleton, $|\mathcal{B}| = 1$. Under A4, the primary-bank attribute $b(\cdot)$ collapses to the unique bank for every firm and every household, the bank-ownership vector $\boldsymbol{S}_h^B$ reduces to a scalar S_{hb_0} for the unique $b_0 \in \mathcal{B}$, and the sum over banks in equation (19) reduces to a single term. Restriction A4 is the financial-sector analogue of A2: relaxing it produces a multi-bank economy with a non-trivial bank–firm matching protocol, which is the natural target of subsequent terms of the sequence.
- **A5.** *Homogeneous banks.* All banks share identical individual parameters: $r_b = r$, $T_b = T$, $\theta_b = \theta$ for every $b \in \mathcal{B}$. Restriction A5 is the financial-sector analogue of A1 (homogeneous households) and is logically independent of A4: under A4 the set $\mathcal{B}$ is a singleton and A5 is vacuous, but in the multi-bank setting the two restrictions become substantively distinct. The current implementation enforces A5 through the use of single-valued parameter vectors in the configuration file, so the question of bank-side heterogeneity does not arise in the present analysis.

The single-government and single-foreign-sector structure of the model, by contrast, is part of the architecture of this term of the sequence rather than a restriction to be relaxed: a multi-level government and a multi-region foreign sector are natural extensions of the framework but are not in the immediate sequence.

4.3 Behavioural restrictions: B-conditions

The behavioural conditions constrain the decision rules of the agents without altering the structural multiplicity. We retain B1–B4 of MAIO1 in their MAIO2-extended form, and introduce one new restriction, B5, that concerns wage formation.

- **B1.** *Fixed prices.* The markup adjustment sensitivity satisfies $s_i = 0$ in equation (16), so the posted price of each KAU is constant at $p_i = (1+\mu_i)\,uc_i$, with uc_i the unit cost defined in equation (14).

- **B2.** *Naive expectations.* Each KAU sets the expected demand to the previous-period realised demand, $q^{De}_{k,t} = q^{D}_{k,t-1}$, with no inertia term.
- **B3.** *Zero target adjustment speed.* The adjustment speeds of stock and capital targets in equations (13)–(12) are set to zero, $v_i = 0$ and $\rho_i = 0$, so inventory and capital deviations from their targets are not corrected within the period and stocks evolve only through depreciation and replacement of intermediate inputs.
- **B4.** *No active counterparty switching.* In the goods market, the preferred-supplier list of each buyer is invariant unless rationing exceeds the switching threshold; in the benchmark of Section 5 no rationing occurs, so no switching takes place. In the labour market, the matching protocol of Section 3.2 is by construction random and one-shot, so no notion of preferred employer is defined. In the credit market, restriction A4 (single bank) makes the question vacuous.
- **B5.** *Exogenous wage offer.* The wage offer w_k of each KAU is fixed at calibration and is constant over the simulation; there is no within-period adjustment of the wage in response to vacancy or unemployment, and no bargaining at the time of matching (Section 3.2). Wage heterogeneity across KAUs is accepted as exogenous when calibrated from sectoral wage data; otherwise the wage vector is uniform.

B1–B5 act on the decision rules of the productive and the labour-supplying agents only; the financial agents (Firm, Bank) and the public agent (Government) have no further behavioural restrictions of their own beyond those already implicit in the descriptions of Sections 2.2–2.5.

4.4 Dynamic regimes: C-conditions

The dynamic-regime conditions identify the macroscopic regime in which the model operates. We retain C1–C2 of MAIO1 and introduce three new conditions, C3–C5, that concern the credit, the foreign, and the fiscal channel respectively.

- **C1.** *No input shortage.* Each KAU can satisfy its planned input use $C_{ji}\,\hat{q}_{k,t}$ from the sum of its current stock and the goods-market purchases of the period. In the benchmark of Section 5, this is a property of the steady state of the input–output system.
- **C2.** *No rationing in the goods market.* Supply meets demand in every transaction of the goods-market cascade of Section 3.1; consequently the purchasing network is invariant under B4, and the realised bilateral allocation $\Gamma_{jk,t}$ equals the preferred-supplier allocation S_{jk}.
- **C3.** *Credit-active regime.* The lending threshold of every bank is set to 1, $\theta_b = 1$ for all $b \in \mathcal{B}$ (Section 2.4), so every firm receives the full amount it requests at phase 4 of the period. The credit-inactive regime $\theta_b = 0$ for all b is a non-vacuous alternative regime, in which the firm structure collapses to the MAIO1 specification.
- **C4.** *Stationary foreign sector.* The growth rate of foreign nominal demand for domestic goods is zero, $g^R = 0$, so $Y^R_t = Y^R$ is constant, and the import price vector $\boldsymbol{p}^R$ is held fixed at its calibrated value (Section 2.6).
- **C5.** *Stationary fiscal stance.* The government's tax rates (τ^V_i, τ^C), transfer parameters (ϕ^T, ϕ^U), and public-consumption shares α^G_i are constant over time (Section 2.5); the government does not borrow from the bank in this term, so its liquidity follows the balanced flow account introduced in Section 2.5.

Remark (Mapping to the benchmark of Section 5). The benchmark configuration of Section 5 is the conjunction A1 $\wedge$ A2 $\wedge$ A3 $\wedge$ A4 $\wedge$ A5 of the agent-configuration conditions, B1–B5 of the behavioural conditions, and C1–C5 of the dynamic regimes. Under this conjunction the model reduces to a deterministic, time-invariant input–output system to which the existence and uniqueness results of Section 5 apply. Each individual restriction can be relaxed independently, generating a distinct extension of the model within the same architectural framework. Relaxing A1 produces the heterogeneous-household submodel of Section 5.6; relaxing A2 generates a sector with competing KAUs and a non-degenerate purchasing network; relaxing A4 produces a multi-bank economy with a non-trivial bank–firm matching protocol; relaxing A5 introduces parameter heterogeneity across banks (interest rates, lending thresholds, loan maturities), which becomes substantive only jointly with the relaxation of A4; relaxing B1 activates endogenous markups; relaxing B3 activates inventory cycles; relaxing B5 calls for an endogenous wage rule; relaxing C1 or C2 introduces real-side rationing; relaxing C3 yields the credit-constrained regime; relaxing C4 introduces foreign-side dynamics relevant to the energy transition; relaxing C5 supports fiscal policy experiments. The order in which these extensions are taken up is the substantive content of the sequence of models announced in Section 1.

5 Steady state of the benchmark

This section derives a closed-form description of the steady state of MAIO2 under a tractable benchmark configuration. We first state the restrictions defining the benchmark (Section 5.1), then characterise capital and labour stocks (Section 5.2), sectoral nominal production (Section 5.3), and the macroeconomic closure that pins down the level of activity (Section 5.4). Section 5.5 discusses existence and uniqueness via the Perron–Frobenius theorem. Section 5.6 extends the construction to heterogeneous household groups, generalising the steady-state characterisation of MAIO1 [1, §2.3.1].

5.1 Benchmark restrictions

Assumption 1 (Benchmark)**.** The benchmark configuration is defined as the conjunction of the agent-configuration, behavioural, and dynamic-regime conditions introduced in Section 4:

(i) **Agent configurations: A1, A2, A3, A4, A5.** Households are homogeneous in individual and relational parameters (A1, Section 4.2); each sector hosts exactly one KAU (A2); each firm owns exactly one KAU (A3); the financial sector is reduced to a single bank (A4); banks are homogeneous in their individual parameters (A5). Restriction A3 is logically independent of A2 and is adopted as a notational convenience: under A1, the firm-to-KAU partition is macroeconomically inert at the benchmark (Section 4.2). Under A2 combined with A3, we use the sectoral index i interchangeably as a KAU index throughout this section. Restriction A5 is vacuous under A4 and becomes substantive only when the financial sector is extended to multiple banks. We relax A1 in Section 5.6.

(ii) **Behavioural restrictions: B1, B2, B3, B4, B5** (Section 4.3). In particular B1 (fixed prices) fixes the posted price of each KAU at $p_i = (1 + \mu_i)\, uc_i$, with uc_i evaluated at the constant wage w_i (B5) and at constant input prices.

(iii) **Dynamic regimes: C1, C2, C3, C4, C5** (Section 4.4). In particular C1–C2 enforce the unconstrained regime in the goods market (no input shortage and no quantity rationing), C3 enforces the credit-active regime $\theta_b = 1$ for all $b \in \mathcal{B}$ (which, under A4, reduces to the single value $\theta_{b_0} = 1$), C4 enforces constant foreign nominal demand $Y_t^R = Y^R$ at constant import prices $\boldsymbol{p}^R$, and C5 enforces a stationary fiscal stance with constant tax rates (τ_i^V, τ^C), transfer parameters (ϕ^T, ϕ^U), and public-consumption shares α_i^G.

Several restrictions in Assumption 1 are *operationally inert* at the steady state: they impose conditions on terms that vanish identically at the fixed point, so the steady-state aggregates they produce coincide with those of any decision rule that converges to the same limit. These restrictions can therefore be replaced, ex post, by the weaker requirement that the model converges to a steady state. The remaining restrictions of Assumption 1—B1, B5, and the C-conditions—are *binding*, in the sense that they determine the structural form of the fixed point itself: their relaxation would alter the steady state, not just the path leading to it.

We illustrate the inert restrictions on three concrete cases. Restriction B2 (naive expectations) imposes $q^{De}_{i,t} = q^{D}_{i,t-1}$; at the fixed point the previous-period demand equals the current one, so expectation errors vanish, $q^{De}_{i*} = q^{D}_{i*}$, regardless of the expectation rule. The production plan equals the realised demand augmented by the own-product use as input, $\hat{q}_{i*} = q^{D}_{i*}/(1-C_{ii})$. Restriction B3 (zero target adjustment speed) deactivates the inventory and capital target corrections in equations (13)–(12); at the fixed point inventory and capital are at their targets, so the adjustment terms $v_i(I^T - I)$ and $\rho_i(\hat{K} - K)$ vanish for any $v_i, \rho_i \geq 0$, and the investment demand reduces to the depreciation flow alone, $\Delta K^{d}_{ji*} = \delta_{ji}\, K_{ji*}$. Restriction B4 (no active counterparty switching) is vacuous under A2, since each sector hosts a single domestic supplier (see the Remark following Section 3.1), and is further inert under C2 (no rationing), which removes the trigger of the switching mechanism.

The macroscopic closure of the model then follows from the binding restrictions C4 and C5: the trade balance closes, $L^R_t = L^R$ constant, so imports equal exports; the government liquidity stabilises, $L^G_t = L^G$ constant, so tax revenue equals the sum of transfers and public consumption.

5.2 Capital and labour at the steady state

At the steady state, capital stocks are at their target. From equation (11), evaluated at the steady-state gross output $\hat{q}_{i*}$ and combined with the identity $\hat{q}_{i*} = q^{D}_{i*}/(1 - C_{ii})$, we obtain the analogue of equation (1) of the working draft of Nieddu:

$$K_{ji*} \;=\; \frac{\gamma_{ji}}{\hat{c}_{ji}}\,\hat{q}_{i*} \;=\; \frac{\gamma_{ji}}{(1 - C_{ii})\,\hat{c}_{ji}}\,q^{D}_{i*}. \tag{37}$$

The investment demand at the steady state is therefore the depreciation flow,

$$\Delta K^{d}_{ji*} \;=\; \delta_{ji}\,K_{ji*} \;=\; \frac{\delta_{ji}\,\gamma_{ji}}{(1 - C_{ii})\,\hat{c}_{ji}}\,q^{D}_{i*}, \tag{38}$$

which is the gross capital expenditure required to keep the capital stock constant. The workforce satisfies

$$N_{i*} \;=\; C_{wi}\,q_{i*} \;=\; \frac{C_{wi}}{1 - C_{ii}}\,q^{D}_{i*}. \tag{39}$$

5.3 Sectoral nominal production

We now derive the input–output system that determines sectoral nominal production $x_{i*} = p_i\,q_{i*}$ at the steady state, where q_{i*} is the gross physical output of sector i. The mapping between gross output and the demand received in the market is the steady-state identity $q^{D}_{i*} = (1 - C_{ii})\,q_{i*}$, which follows from the balance between production and own-product use as intermediate input. Two matrices play a central role.

Definition 2. The nominal technical-coefficient matrix $\boldsymbol{A} = (A_{ij})$ collects the input–output linkages between sectors at constant prices,

$$A_{ij} \;=\; p_i\,C_{ij}\,p_j^{-1}. \tag{40}$$

The nominal capital-demand intensity matrix $\boldsymbol{\Lambda} = (\Lambda_{ij})$ collects the steady-state capital flow that sector j absorbs from sector i as a fraction of its output,

$$\Lambda_{ij} \;=\; \frac{\delta_{ij}}{\hat{c}_{ij}}\,p_i\,\gamma_{ij}\,p_j^{-1}. \tag{41}$$

Equation (41) is read as follows: $\gamma_{ij}\, q_j$ is the desired capital stock of input i in sector j at unit utilisation (numerator of (37) dropping the $1/(1-C_{ii})$ factor); δ_{ij} is the period depreciation; and $\hat{c}_{ij}$ is the target utilisation. The product $\delta_{ij}\gamma_{ij}/\hat{c}_{ij}$ is therefore the period flow of capital good i required by sector j to keep its capital stock at target, expressed per unit of physical output of j. The price conversion $p_i p_j^{-1}$ turns this into a nominal share.

Trade with the rest of the world is summarised by three vectors of import shares. We denote by $\Gamma^i_{Rj} \in [0,1]$ the share of sector j's nominal demand for product i (intermediate input plus capital good) that is satisfied by foreign supply, and by $\Gamma^i_{RH}, \Gamma^i_{RG} \in [0,1]$ the analogous shares for households and government. These shares are the nominal counterparts of the import quotas read from the implementation: in particular, $\Gamma^i_{RG} = 1 - q^G$ when government purchases of product i are subject to the same import quota q^G for every sector. The *domestic* shares are $1 - \Gamma^i_{R\bullet}$.

Proposition 3 (Sectoral nominal production at the steady state). *Under Assumption 1, the steady-state vector of sectoral nominal productions $\boldsymbol{x}_* = (x_{1*}, \ldots, x_{N_A*})^\top$ solves the linear system*

$$x_{i*} \;=\; \sum_{j=1}^{N_A} (1-\Gamma^i_{Rj})\,(A_{ij}+\Lambda_{ij})\,x_{j*} \;+\; (1-\Gamma^i_{RH})\,\alpha_{iH}\,Y^D_* \;+\; (1-\Gamma^i_{RG})\,\alpha_{iG}\,C^G_* \;+\; \alpha^R_i\,c^R Y^R, \tag{42}$$

where Y^D_* is aggregate household disposable income, C^G_* is aggregate government consumption, and the last term is the foreign demand for domestic exports.

The right-hand side of (42) expresses the demand received by sector i as the sum of four components, each expressed as a domestic share of a total demand: the intermediate-input demand of domestic sectors j, $A_{ij}\,x_{j*}$; the capital-replacement demand of domestic sectors, $\Lambda_{ij}\,x_{j*}$; the household consumption, $\alpha_{iH}\,Y^D_*$, where α_{iH} denotes the (common) expenditure share of households on sector i; the government consumption, $\alpha_{iG}\,C^G_*$; and the foreign demand on domestic exports, $\alpha^R_i\,c^R Y^R$. Equation (42) reproduces equation (2) of the internal working note of Nieddu and is derived from the micro-level market-clearing identities in Appendix A.

In compact form, defining the augmented domestic-coefficient matrix $\widetilde{\boldsymbol{M}} = (\widetilde{M}_{ij})$ with elements

$$\widetilde{M}_{ij} \;=\; (1-\Gamma^i_{Rj})\,(A_{ij}+\Lambda_{ij}), \tag{43}$$

equation (42) reads

$$\boldsymbol{x}_* \;=\; \widetilde{\boldsymbol{M}}\,\boldsymbol{x}_* + \boldsymbol{f}_*, \quad f_{i*} \;=\; (1-\Gamma^i_{RH})\,\alpha_{iH}\,Y^D_* + (1-\Gamma^i_{RG})\,\alpha_{iG}\,C^G_* + \alpha^R_i c^R Y^R. \tag{44}$$

When the spectral radius of $\widetilde{\boldsymbol{M}}$ satisfies $\rho(\widetilde{\boldsymbol{M}}) < 1$, the generalised Leontief inverse $\widetilde{\boldsymbol{L}} = (\boldsymbol{I} - \widetilde{\boldsymbol{M}})^{-1}$ exists and is non-negative, and we can write

$$\boldsymbol{x}_* \;=\; \widetilde{\boldsymbol{L}}\,\boldsymbol{f}_*. \tag{45}$$

Equation (45) reduces to the steady-state production $\boldsymbol{x}_* = \boldsymbol{L}\,\boldsymbol{\alpha}\,Y^*_H$ of MAIO1 (equation (32) therein) under three additional restrictions: no capital ($\boldsymbol{\Lambda} = \boldsymbol{0}$), no foreign trade ($\Gamma^i_{R\bullet} = 0$, $Y^R = 0$), and no government ($C^G_* = 0$). Equation (45) therefore generalises that benchmark by adding three structural sources of demand: capital replacement, government consumption, and foreign demand.

5.4 Macroeconomic closure: disposable income, fiscal balance, trade balance

Equation (45) expresses sectoral production *conditional on* the level of disposable income Y^D_* and on government consumption C^G_*. To pin down the absolute level of activity, we need three additional steady-state conditions: the definition of disposable income, the government's flow-consistency condition, and the trade balance. We derive them under the homogeneous restriction A1 of Assumption 1; the heterogeneous extension is in Section 5.6.

Disposable income. Aggregate disposable income of households at the steady state is the sum of dividends and wages, net of personal taxes (zero in the present specification, as taxation falls on KAUs only) plus government transfers:

$$Y_*^D \;=\; \Pi_*^H \;+\; W_* \;+\; T_* \;=\; \sum_i \pi_{i*} \;+\; \sum_i w_i\, N_{i*} \;+\; N_H\, \phi^T\, \bar{w}_*. \tag{46}$$

Here π_{i*} is the steady-state net profit of KAU i, $W_* = \sum_i w_i\, N_{i*}$ is the aggregate wage bill, $\bar{w}_*$ is the average wage of the economy, and $N_H \phi^T \bar{w}_*$ is the aggregate transfer (unemployment benefits vanish in the no-rationing benchmark, since labour demand equals supply by construction). Net profits at the steady state equal revenues net of value-added tax, intermediate-input costs, the wage bill, and depreciation, multiplied by one minus the corporate tax rate:

$$\pi_{i*} \;=\; (1-\tau^C)\,\big[\,x_{i*} \;-\; T_{i*}^V \;-\; \mathrm{Cost}_{i*}^{\mathrm{int}} \;-\; W_{i*} \;-\; \mathcal{D}_{i*}\,\big]^+, \tag{47}$$

where the bracket components are: the value-added tax $T_{i*}^V = \tau_i^V/(1+\tau_i^V)\, x_{i*}$ on the gross sectoral revenue x_{i*}; the intermediate-input cost $\mathrm{Cost}_{i*}^{\mathrm{int}} = \sum_{j\neq i} A_{ji}\, x_{i*}$ (the term $j = i$ is excluded because own-product use is internal and is not transacted); the wage bill $W_{i*} = w_i\, N_{i*}$; and the depreciation expense $\mathcal{D}_{i*} = \sum_j \Lambda_{ji}\, x_{i*}$ associated with the period-t utilisation of the capital stock.[2] The structure of (47) reflects the implementation: in the code `LocalKAU_ordinato.py`, method `compute_earnings`, the corporate tax is levied on net earnings (revenues minus VAT, intermediate costs, wages, and depreciation), and profits accruing to firms are net earnings less the corporate tax. The non-negativity operator $(\cdot)^+$ implements the implementation rule that profits and the corresponding tax base are floored at zero.

Government flow consistency. At the steady state, government liquidity is constant, $L_t^G = L_*^G$. The flow-consistency condition then requires that taxes received equal the sum of transfers paid and consumption,

$$\mathcal{T}_* \;=\; T_* \;+\; C_*^G, \quad \text{where} \quad \mathcal{T}_* \;=\; \sum_i \big(T_{i*}^V + T_{i*}^C\big), \tag{48}$$

with $T_{i*}^V = \tau_i^V/(1+\tau_i^V)\, x_{i*}$ and $T_{i*}^C = \tau^C\,$(net earnings of i). Equation (48) implicitly determines C_*^G in terms of the production vector $\boldsymbol{x}_*$. Together with the budget allocation rule $B_{i*}^G = \alpha_i^G\, C_*^G$ (Section 2.5, equation (23)), this closes the government block.

Trade balance. The third closure condition is the trade balance, equivalent to the constancy of the foreign liquidity L_*^R:

$$\underbrace{\sum_i \alpha_i^R\, c^R Y^R}_{\text{exports}} \;=\; \underbrace{\sum_{ij} \Gamma_{Rj}^i\,(A_{ij} + \Lambda_{ij})\, x_{j*} + \sum_i \Gamma_{RH}^i\, \alpha_{iH}\, Y_*^D + \sum_i \Gamma_{RG}^i\, \alpha_{iG}\, C_*^G}_{\text{imports}}\,. \tag{49}$$

Equation (49) is not an independent equation: it follows from (42) by summing across sectors and using $\sum_i x_{i*} = \sum_i (1 - \Gamma_{R\bullet}^i)$ (total nominal demand on i) + exports. We record it here because it makes the open-economy interpretation transparent and because it is the natural numerical consistency check in simulations.

[2] The depreciation expense $\mathcal{D}_{i*}$ that appears on the cost side of sector i is matched, at the steady state, by the depreciation revenue $\sum_j \Lambda_{ij}\, x_{j*}$ accruing to sector i as a capital-good supplier. Aggregating across the economy, the depreciation flow appears once on the revenue side and once on the cost side; it is therefore neutral with respect to aggregate profits.

5.5 Existence, uniqueness, and the level of activity

Equations (42)–(48) jointly determine the steady-state vector of sectoral productions $\boldsymbol{x}_*$ and the aggregate scalars (Y_*^D, C_*^G). Existence and uniqueness can be established under a Perron–Frobenius-type spectral radius condition on the augmented matrix obtained by combining (42) and the linear closure of Y_*^D and C_*^G in $\boldsymbol{x}_*$.

Proposition 4 (Existence and uniqueness of the benchmark steady state). *Suppose Assumption 1 holds, all parameters are non-negative, the matrices $\boldsymbol{A}$, $\boldsymbol{\Lambda}$, $\widetilde{\boldsymbol{M}}$ are non-negative, and the spectral radius of the augmented system matrix obtained by stacking* (42)*, the linearisation of* (46) *in $\boldsymbol{x}_*$, and the linearisation of* (48) *in $\boldsymbol{x}_*$ is strictly less than one. Then the steady state $(\boldsymbol{x}_*, Y_*^D, C_*^G)$ exists, is unique, and has non-negative components, and is the global limit of trajectories of the unconstrained dynamics from any non-negative initial condition.*

The proof is given in Appendix B. The argument follows the strategy of MAIO1: under the unconstrained regime (no rationing in any market), the dynamics of (42)–(48) reduce to a non-negative linear autonomous system; the Perron–Frobenius theorem then delivers existence, uniqueness, and global stability of the fixed point. The spectral-radius condition imposes that the combined leakages from intermediate inputs, capital depreciation, foreign trade, household saving, and net government extraction are sufficient to dampen the intersectoral demand multiplier.

5.6 Heterogeneous households

We now relax restriction A1 of Assumption 1 and extend the steady-state characterisation to the case of $G \geq 2$ heterogeneous household groups. Each group $g \in \{1, \ldots, G\}$ contains $N_{H,g}$ households, all sharing the same individual parameters $(c_g^y, c_g^w, \lambda_g^T, \boldsymbol{\alpha}_g)$ and the same group-level ownership structure $\boldsymbol{S}_g = (S_{gf})_{f=1,\ldots,N_F}$. The total number of households is $N_H = \sum_g N_{H,g}$. The aim is to identify the steady-state distribution of income and wealth across groups, and to show that the production system (42) retains its form, with macroscopic propensities expressed as weighted averages of group propensities. The construction follows [1, §2.3.1].

Group-level income shares. Let Y_{g*}^D denote the steady-state aggregate disposable income of group g and define the income share $s_{g*} = Y_{g*}^D / Y_*^D$, with $\sum_g s_{g*} = 1$. From the profit-distribution mechanism of equation (1), combined with the wage-distribution rule, the group-level income at the steady state is a linear function of sectoral profits and the wage bill, which in turn are linear in $\boldsymbol{x}_*$. Tracing this through, the income share vector $\boldsymbol{s}_* = (s_{1*}, \ldots, s_{G*})^\top$ satisfies the fixed-point equation

$$\boldsymbol{s}_* \;=\; \mathcal{A}\, \boldsymbol{s}_* + \boldsymbol{b}, \tag{50}$$

where the matrix $\mathcal{A}$ and the vector $\boldsymbol{b}$ depend on the ownership structure $(\boldsymbol{S}_g)$, the technical coefficient matrix $\boldsymbol{A}$, the capital intensities $\boldsymbol{\Lambda}$, the trade structure $(\Gamma_{R\bullet}^i)$, the group consumption shares $(\boldsymbol{\alpha}_g)$, and the wage–employment structure. Equation (50) is the analogue of equation (48) of MAIO1, with two structural extensions: the wage channel adds a labour-income component to the group-level income; and the capital and trade matrices $\boldsymbol{\Lambda}$, $(1 - \Gamma_{R\bullet}^i)$ enter through the production structure that links income to demand. The explicit construction of $\mathcal{A}$ and $\boldsymbol{b}$ is given in Appendix C.

Macroscopic propensities and wealth distribution. At the steady state, each group consumes its full income (savings are zero in flow terms, since wealth is at its target), so the macroscopic expenditure-share vector that enters (42) is the income-weighted average of the group-level vectors:

$$\boldsymbol{\alpha}_{H*} \;=\; \sum_{g=1}^{G} s_{g*}\, \boldsymbol{\alpha}_g. \tag{51}$$

Similarly, the macroscopic wealth-to-income ratio is

$$\lambda^T_{H*} \;=\; \sum_{g=1}^{G} s_{g*}\,\lambda^T_g, \tag{52}$$

which generalises equation (51) of MAIO1. The group-level wealth satisfies

$$M^H_{g*} \;=\; \lambda^T_g\, s_{g*}\, Y^D_*, \qquad \frac{M^H_{g*}}{M^H_*} \;=\; \frac{\lambda^T_g\, s_{g*}}{\lambda^T_{H*}}, \tag{53}$$

which generalises equations (49) and (52) of MAIO1.

Remark. The structural form of the production system (42) is unchanged in the heterogeneous case: only the macroscopic propensities $\boldsymbol{\alpha}_H, \lambda^T_H$ are reinterpreted as weighted averages through equations (51)–(52). This separation between intersectoral structure and household distribution is the formal basis for the indistinguishability discussion of Section 6: the steady-state aggregates of the heterogeneous economy coincide with those of a representative-household economy if and only if $(\boldsymbol{\alpha}_g, \lambda^T_g)$ are uncorrelated with the steady-state income shares s_{g*}. This is the MAIO2 analogue of the indistinguishability conditions discussed in [1, §2.3].

6 Aggregation conditions for MAIO2

The steady-state characterisation of Section 5 treats the macroscopic input–output system and the heterogeneous-household distribution as two coupled but separable blocks (equations (42), (50)–(52)). This section provides the theoretical underpinning of that separation. We restate the three indistinguishability properties of MAIO1 [1, §2.3] in the MAIO2 setting (Section 6.1), show that the three new institutional agents—bank, government, rest of the world—are compatible with those properties by construction (Section 6.2), discuss the productive-unit side under the agent-configuration restriction A2 and indicate what changes when A2 is relaxed (Section 6.3), and assess the role of the two new decentralised markets, labour and credit (Section 6.4). We conclude with the formal statement of the equivalent-household representation of MAIO2 at the benchmark (Section 6.5).

6.1 Household indistinguishability extended

We organise households into $G \geq 1$ groups, with $N_{H,g}$ households per group and total population $N_H = \sum_g N_{H,g}$. Group g is characterised by the parameter tuple $(c^y_g, c^w_g, \lambda^T_g, \boldsymbol{\alpha}_g)$ already introduced in Section 5.6, and by the group-level ownership vector $\boldsymbol{S}_g = (S_{gf})_{f=1,\ldots,N_F}$. Within a group, every household is identical; across groups, parameters may differ.

The three indistinguishability properties of MAIO1 read, in the MAIO2 setting:

- **P1 (individual indistinguishability).** All groups share identical individual parameters: $c^y_g = c^y$, $c^w_g = c^w$, $\lambda^T_g = \lambda^T$, $\boldsymbol{\alpha}_g = \boldsymbol{\alpha}$ for every g. Equivalently, $G = 1$ up to relabelling. Under P1 every macroscopic propensity that aggregates household parameters reduces to the common individual value, and the macroscopic-household description coincides *trivially* with the microscopic one.
- **P2 (weak relational indistinguishability).** All groups share identical relational parameters with the productive sector: the ownership share of group g in firm f is the same for every group up to the group size, $S_{gf} = N_{H,g}/N_H$. Equivalently, every firm is owned by the household population uniformly per capita. Under P2 the steady-state income share of group g is determined independently of $(\boldsymbol{\alpha}_g, \lambda^T_g)$, and equals the population share, $s_{g*} = N_{H,g}/N_H$.

- **P3 (strong relational indistinguishability).** P2 holds, and the remaining parameters that enter macroscopic propensities through equations (51) and (52), namely $(\boldsymbol{\alpha}_g, \lambda_g^T)$, are identical across groups. Under P3, the macroscopic propensities $\boldsymbol{\alpha}_{H*}, \lambda_{H*}^T$ are exactly the common group values, and the macroscopic steady state is qualitatively identical to that of a single representative household.

The natural inclusions P1 $\Rightarrow$ P3 $\Rightarrow$ P2 hold by construction. We refer to the parameters that remain constant macroscopically under P2 (e.g., the wealth-to-income ratio λ_H^T if every group has the same λ^T) as *macro-invariant*, in the terminology of [1, §2.3].

The new parameter of MAIO2: the propensity to consume transfers. Compared to MAIO1, the household parameter list of MAIO2 includes the propensity to consume out of transfers λ_g^T (Section 2.1). Equation (52) shows that λ_g^T enters the macroscopic wealth-to-income ratio through the same weighted-average mechanism that governs c_g^y and $\boldsymbol{\alpha}_g$ in MAIO1: it is therefore a candidate for macro-invariance under P2, and is exactly macro-invariant under P3. No new symmetry beyond those of P1–P3 is required to handle the new parameter.

6.2 Bank, government, and the foreign sector as aggregation-compatible institutions

The three institutional agents of MAIO2 enter the household income through three additional channels: bank dividends, government transfers, and the foreign trade balance. Each channel introduces *de jure* an additional ownership-like relational structure between the household population and the institutional agent. We show below that the specifications adopted in Section 2.4, Section 2.5, and Section 2.6 are constructed so that this relational structure is uniform across households by design and does not impose new symmetry conditions beyond P1–P3.

Bank ownership shares and the dividend distribution. The bank-dividend channel of MAIO2 is governed by the vector of bank ownership shares $\boldsymbol{S}_h^B = (S_{hb})_{b\in\mathcal{B}}$ introduced in Section 2.1: the dividend received by household h from the multi-bank financial sector is $\Pi_{h,t}^B = \sum_b S_{hb}\,\Pi_{b,t}^B$ (equation (19)). Under A4 (single bank) the sum reduces to a single term involving the unique bank b_0, with share S_{hb_0}. The current implementation calibrates the bank ownership shares to the uniform specialisation $S_{hb} = 1/N_H$ for every (h, b): under this specialisation, every group receives a bank-dividend income proportional to its population share, $N_{H,g}/N_H$, exactly as required by P2 for the firm-ownership channel. The bank therefore behaves, for aggregation purposes, as an additional "firm" uniformly owned per capita by the household population, and the dividend distribution channel introduces no new symmetry condition beyond P1–P3.

Remark (Counterfactual: proportional-to-deposits dividends). A natural alternative rule would distribute bank dividends in proportion to households' deposits, replacing the uniform calibration $S_{hb} = 1/N_H$ with shares $S_{hb} \propto D_{h,t}$. Under this counterfactual, the ownership share $S_{hb,t}$ would become endogenous, depending on the steady-state wealth distribution M_{g*}^H/M_*^H given by (53). The compatibility with P2 would then require the additional condition $M_{g*}^H/M_*^H = N_{H,g}/N_H$, which is in general violated unless λ_g^T and s_{g*} are perfectly correlated. The uniform calibration adopted in Section 2.4 avoids this complication and is, in this sense, the maximally aggregation-friendly specification.

Government transfers and the implicit "fiscal ownership". The lump-sum transfer mechanism of the government (Section 2.5) is uniform across households: every household receives the same per-capita transfer $T_{h,t} = \phi^T\,\bar{w}_t$, proportional to the economy-wide average wage $\bar{w}_t$ (equation (22)), and every unemployed household receives the same unemployment benefit $U_{h,t} = \phi^U\,\bar{w}_t$. The transfer channel therefore induces an implicit "fiscal ownership" share

$\tilde{s}_h^G = 1/N_H$, uniform across households, analogous to the uniform bank-ownership calibration $S_{hb} = 1/N_H$ discussed above. The unemployment benefit channel, however, operates only on the unemployed sub-population: $\tilde{s}_{h,t}^U = (N_H - N_{e,t})^{-1}$ if household h is unemployed at time t, and zero otherwise. At the benchmark steady-state of Section 5, full employment holds ($N_{e,*} = N_H$), so $\tilde{s}_{h,*}^U = 0$ for every household and the unemployment-benefit channel is inert. Outside the benchmark, the unemployment-benefit channel introduces an additional source of ownership-like inhomogeneity that is not captured by P1–P3.

Sectoral tax rates τ_i^V and the corporate tax rate τ^C act on the productive sector, not directly on households: they reduce the sector-level distributable profit $\pi_{f,*}^F$, which is then redistributed across households through the firm-ownership channel S_{gf}. Tax rates therefore enter the macroscopic system as parameters of the production block (42) and require no new household-side symmetry.

The foreign sector and the exogeneity of foreign demand. The rest-of-the-world (Section 2.6) interacts with the domestic economy through the sectoral foreign demand $\boldsymbol{Y}^R$ and the calibrated export weights $(\omega_{ki}^R)_{k\in\mathcal{K}_i}$ on the supply side (equation (27)), and through the import quotas $q_b^R(i)$ on the demand side (Section 3.1). Under condition C4 of Section 4.4 (stationary foreign sector), $\boldsymbol{Y}^R$ and $\boldsymbol{p}^R$ are constant and exogenous; under restriction A2, the export weights collapse to $\omega_{ki}^R = 1$ for the unique producer k of each sector i, and the import quotas $q_b^R(i)$ are exogenous parameters of the buyer-side specification. The foreign sector therefore enters the macroscopic input–output system as an external demand component, with no distributive impact on the household population beyond that mediated by sectoral profits.

6.3 Productive units: indistinguishability under A2 and beyond

Under restriction A2 of Assumption 1 (one KAU per sector), the productive-unit side of the model is automatically *exactly aggregated* at the sectoral level: the macroscopic parameters $(\mu_i, C_{ji}, C_{wi}, \delta_{ji}, \hat{c}_{ji}, w_i)$ are defined directly at the sector index and do not require further symmetrisation. The aggregation issues that arise on the household side under P1–P3 do not have an immediate counterpart on the productive side under A2.

The question becomes substantive when A2 is relaxed, that is, when sector i hosts $|\mathcal{K}_i| \geq 2$ distinct KAUs. In that case, the construction of a representative KAU for sector i requires an additional indistinguishability condition, that we label $\mathbf{P^K}$, on the KAU parameter vector. The natural analogue of P1 for the productive side is:

- $\mathbf{P_1^K}$ *(individual KAU indistinguishability).* All KAUs of a given sector share identical individual parameters: $\mu_{k(i)} = \mu_i$, $C_{j,k(i)} = C_{ji}$, $C_{w,k(i)} = C_{wi}$, $\delta_{j,k(i)} = \delta_{ji}$, $\hat{c}_{j,k(i)} = \hat{c}_{ji}$, $w_{k(i)} = w_i$ for every $k \in \mathcal{K}_i$.

Under $\mathbf{P_1^K}$, the macroscopic sector-i parameters are simply the common KAU values, and the steady-state input–output system (42) retains its form with sectoral indices replaced by KAU-sectoral indices. The analogues of P2 and P3 for the productive side (uniform purchasing-network shares and uniform labour-allocation shares within a sector) extend naturally, in parallel with the household discussion of Section 6.1; we do not pursue them further in this term of the sequence, since the benchmark of Section 5 imposes A2 and the question is vacuous.

6.4 The decentralised markets and the limits of exact aggregation

The three decentralised markets of MAIO2 affect aggregation through the relational networks that they generate or perturb. We discuss each in turn under the benchmark restrictions A2 and C1–C3, where the discussion is constructive, and indicate where the analogy with the goods market of MAIO1 breaks the analytical tractability when those restrictions are relaxed.

Goods market. Under A2 the goods market is degenerate as discussed in the Remark following Section 3.1: every commodity has a single domestic supplier, and the purchasing network is a star graph. The realised bilateral allocation matrix $\Gamma_{jk,t}$ of MAIO1 collapses to a deterministic incidence matrix, and the symmetry condition $\Gamma_{jk,t} = S_{jk}$ that defines P3 for the goods market is automatically satisfied. The decentralised protocol of Section 3.1 therefore contributes no obstruction to P3 at the benchmark.

When A2 is relaxed (the next term of the sequence), the goods market recovers the analytical features of MAIO1: in the unconstrained regime $\Gamma = S$ continues to hold and P3 is preserved; under rationing, $\Gamma_{jk,t}$ becomes endogenous and time-dependent, and exact P3 aggregation breaks in the same way it does in [1, §2.3.4].

Labour market. The random matching protocol of Section 3.2 induces, at each period, an employment incidence matrix $E_{h,k,t} \in \{0,1\}$ that maps household h to its employer k (with $E_{h,k,t} = 0$ for unemployed households). The aggregation-relevant object is the expected employment share $\sigma^N_{g,k,t} = \mathbb{E}[E_{h\in g,k,t}]$, which describes the share of group-g households employed by KAU k. Under C2–C3 (no goods-market rationing, credit-active regime) the steady-state labour demand equals the steady-state labour supply (full employment), and the random matching distributes group-g households across KAUs in proportion to the labour-demand share of each KAU,

$$\sigma^N_{g,k,*} = \frac{\hat{N}_{k,*}}{\sum_{k'} \hat{N}_{k',*}}, \qquad k \in \mathcal{K}, \tag{54}$$

independently of g. The employment matrix is therefore *symmetric across groups* at the benchmark steady state, and P3 is preserved: the labour income received by group g is its population share times the aggregate wage bill.

When the labour market exits the full-employment regime (e.g., under negative shocks that activate the firing rule of (29)), the random matching distributes unemployment uniformly across groups in expectation, but realised unemployment correlates with past labour demand and breaks the exact group-symmetry of (54) in finite samples. P3 then holds only in mean: the equivalent-household description captures the expected aggregate dynamics, with sample-driven deviations that vanish in the infinite-population limit.

Credit market. Under restriction A4 (single bank, Section 4.2) the credit network is trivial: every firm is linked to the unique bank, and the credit-market analogue of P2 (uniform credit-network shares) is automatically satisfied. Under condition C3 the granting rule is non-binding, and the credit-market protocol contributes no obstruction to P3.

6.5 Statement: the equivalent-household economy of MAIO2 at the benchmark

Combining the four results above, the aggregation theory of MAIO2 at the benchmark of Assumption 1 can be summarised in a single statement. We use the notation of Section 5.6: $\boldsymbol{x}_*$ is the steady-state vector of nominal sectoral production, Y^D_* the aggregate disposable income, $(\boldsymbol{\alpha}_{H*}, \lambda^T_{H*})$ the macroscopic propensities defined in (51)–(52).

Proposition 5 (Equivalent-household representation)**.** *Under Assumption 1, the steady-state aggregates $(\boldsymbol{x}_*, Y^D_*, \boldsymbol{\alpha}_{H*}, \lambda^T_{H*})$ of the multi-agent MAIO2 economy coincide with the steady-state aggregates of a one-household economy with parameters $(c^y_H, c^w_H, \lambda^T_H, \boldsymbol{\alpha}_H)$ if and only if $(\boldsymbol{\alpha}_g, \lambda^T_g)$ are uncorrelated with the income shares s_{g*} across groups, that is,*

$$\sum_{g=1}^{G} s_{g*}\left(\boldsymbol{\alpha}_g - \boldsymbol{\alpha}_{H*}\right) = \mathbf{0}, \qquad \sum_{g=1}^{G} s_{g*}\left(\lambda^T_g - \lambda^T_{H*}\right) = 0. \tag{55}$$

A sufficient condition for (55) *is the strong relational indistinguishability P3 (Section 6.1). The institutional channels of MAIO2 (bank dividends, government transfers, foreign trade) contribute to* $(\boldsymbol{\alpha}_{H*}, \lambda^T_{H*})$ *through aggregation-compatible weights (Section 6.2); the decentralised markets of labour and credit contribute exact aggregation under C2–C3 (Section 6.4).*

The proposition has two readings. As a positive result, it identifies P3 as the unique source of indistinguishability obstructions in MAIO2 at the benchmark: the additional institutional and financial structure introduced in this term of the sequence does not, by itself, add new aggregation constraints. As a negative result, it shows that P3 *remains* the binding condition for exact aggregation, exactly as in MAIO1: enriching the model with bank, government, and a foreign sector does not relax the fundamental indistinguishability requirement on household preferences.

A numerical illustration of Proposition 5, comparing the steady-state aggregates of a multi-group MAIO2 economy with those of its representative-household counterpart as a function of group-level heterogeneity in $(\boldsymbol{\alpha}_g, \lambda^T_g)$ and of the correlation with the steady-state income shares, is left for the extended version of the paper.

7 Discussion

What MAIO2 clarifies relative to MAIO1. The first term of the MAIO sequence [1] is a closed multi-sector economy of households and KAUs, in which production is realised through a decentralised goods market and the rest of the economic apparatus is either absent (bank, government, foreign sector) or held fixed in the background. MAIO2 internalises the institutional and financial structure that the first term took as exogenous, introducing fixed capital and labour as complementary factors of production, a single bank with a loan portfolio and an interest-income flow, a government with sectoral taxation and lump-sum transfers, and a foreign sector with exogenous demand and accommodating supply. Each of these additions enlarges the state of the economy and complicates the period accounting, yet the analytical structure of the first term is preserved: under Assumption 1, the steady-state production system retains the linear input–output form of (42) (Proposition 3); existence and uniqueness of the steady state continue to be governed by a Perron–Frobenius argument (Proposition 4); and the exact-aggregation conditions of MAIO1 extend without strengthening to the new institutional channels (Proposition 5). The methodological payoff of this term is therefore not the introduction of additional realism *per se*, but the demonstration that a tractable institutional enrichment is feasible—that the analytical scaffolding of MAIO1 absorbs the new agents and markets at no qualitative cost in terms of tractability.

Behavioural and institutional simplifications, by design. The price of this tractability is a set of explicit simplifications, collected in the B- and C-conditions of Section 4 and embedded in the implementation of the new markets in Section 3. We highlight the four that are the most substantial candidates for relaxation in subsequent terms of the sequence. First, the labour market is matched by a one-shot random protocol (Section 3.2) with an exogenous wage offer (restriction B5); endogenous wage formation and a preferred-employer search protocol with rationing on both sides would bring the labour market in line with the four-step decentralised structure of the goods market. Second, the credit market is regulated by a binary granting rule administered by a single bank (Section 3.3); credit-worthiness assessment, partial rationing, and competition between multiple banks would lift the single-bank simplification and introduce genuine financial frictions into the model. Third, the markup-adjustment rule of equation (16) is part of the general framework of MAIO2: activating it (i.e., relaxing restriction B1 by setting $s_i \neq 0$ in the parameter file) generates within-period price flexibility and inflation–deflation dynamics, both of which are silent at the benchmark of this paper. Endogenous price setting is therefore not an extension of MAIO2, but a relaxation of a single parameter within MAIO2;

we treat it on the same footing as the relaxation of B3 below. Fourth, the inventory and capital adjustment speeds are set to zero (restriction B3); re-activating them ($v_i, \rho_i > 0$) produces stock-driven cycles that are visible in the goods market and feed back into the production plan through equations (13)–(12). Both B1 and B3 are restrictions that the user of the model can lift simply by changing the parameter file, without modifying the simulation code: this is what we mean, in Section 5, when we say that the benchmark of Assumption 1 is the *benchmark* of a more general framework.

Beyond A2: the productive side. The agent-configuration restriction A2 (one KAU per sector) is the single most consequential structural simplification of this term. Relaxing it produces a sector with $|\mathcal{K}_i| \geq 2$ competing KAUs, an endogenous purchasing network that ceases to be a star graph, and a non-trivial extension of the aggregation theory of Section 6 to the productive side. As anticipated in Section 6.3 and in the note closing Section 6, the natural KAU-side analogue of the household indistinguishability conditions is $\mathbf{P_1^K}$ (identical individual parameters within a sector), with $\mathbf{P_2^K}$ and $\mathbf{P_3^K}$ defined in parallel through purchasing-network and labour-allocation symmetries. The combined multi-KAU, multi-household model recovers the full P/A/C taxonomy of MAIO1 in a setting where labour, capital, credit, and trade flows all operate simultaneously, and is the natural next term of the sequence.

Toward data-driven applications: SAM calibration and the PNIEC scenario. The institutional enrichment of MAIO2 opens the model to data-driven calibration. All the structural parameters that enter the benchmark of Assumption 1—technical coefficients C_{ji}, capital coefficients Λ_{ji}, labour intensities C_{wi}, sectoral markups μ_i, public-consumption shares α_i^G, sectoral tax rates τ_i^V, household consumption shares $\boldsymbol{\alpha}_g$, foreign demand $\boldsymbol{Y}^R$, and the supply weights ω_{ki}^R—can be read directly from a Social Accounting Matrix (SAM) of the economy under study. Once calibrated, the steady-state characterisation of Section 5 provides an analytical benchmark against which the simulated trajectories can be validated, and the aggregation theory of Section 6 provides explicit conditions under which the calibrated model admits an exact representative-household representation. The applicability of this calibration pathway is illustrated, in a companion line of work, by the scenario analysis of the Italian National Integrated Energy and Climate Plan (PNIEC), in which a SAM of the Italian economy is used to evaluate the macroeconomic and distributional implications of the energy transition. The PNIEC analysis is described elsewhere; here we limit ourselves to the observation that MAIO2 supports such an analysis without sacrificing the analytical tractability that motivates the sequence in the first place.

Positioning within the recent ABM–IO literature. A recent and growing body of work has emphasised the data-driven calibration of agent-based input–output models, with applications to network shocks and pandemic propagation [5, 7], and to euro-area macroeconomic analysis through central-bank-developed platforms such as BeforeIT [6]. MAIO2 is complementary to that programme rather than competitive with it: the present paper does not aim at empirical fit, but at an explicit analytical scaffolding on which calibrated extensions can be built. The premium placed by the MAIO programme on tractability, P/A/C taxonomies, and sequence-controlled extensions distinguishes it from the purely data-driven tradition, while the institutional enrichment introduced in this term begins to bridge the two by making SAM-based calibration the natural next step. The Eurace family of models [2, 3], in which several authors of the present paper have participated, offers a different point of reference: a comparable institutional richness with a less aggressive analytical layer. MAIO2 can be read as an attempt to recover, within an Eurace-compatible institutional architecture, the kind of analytical control that the original Eurace contributions did not pursue.

Closing. The value added of the sequential programme of which this paper is the second term is not a single model, but an explicit and controllable path from microscopic complexity to macroscopic description. Each term of the sequence introduces a new layer of structure—households, KAUs, decentralised markets in MAIO1; capital, labour, credit, government, foreign sector in MAIO2—and characterises the conditions under which the new layer is compatible with an analytical description of the aggregate. By making the behavioural and dynamic-regime restrictions explicit, the sequence keeps track, at every step, of what has been assumed away in exchange for tractability. Subsequent terms of the sequence will relax these restrictions one at a time, in the order suggested by the discussion above, and document the analytical or computational cost of each relaxation. We view this discipline as a contribution in its own right, beyond the specific results of any individual term.

A Derivation of the sectoral nominal production system

This appendix derives equation (42) of Proposition 3 from the micro-level decision rules of Section 2. Throughout, we work under Assumption 1; the homogeneous-household restriction is not used in this appendix and the derivation extends without modification to the heterogeneous case of Section 5.6 (in which α_{iH} becomes the macro propensity $\alpha_{iH*} = \sum_g s_{g*}\alpha_{ig}$).

Step 1: Sales equal demand at the steady state. Under no rationing in the goods market, the realised demand of every buyer is fully satisfied. Aggregating the demand received by sector i from all buyer classes, the steady-state physical sales of sector i equal the total physical demand directed to it.

Step 2: Total physical demand decomposition. Five buyer classes generate physical demand for the product of sector i:

(a) **Intermediate input demand** from each sector j: each KAU j requires C_{ij} units of input i per unit of gross output, so the total intermediate-input demand on i is $\sum_j C_{ij}\, q_{j*}$.

(b) **Capital-replacement demand** from each sector j: from (38), the steady-state investment demand for capital good i from sector j is $\delta_{ij}\, \gamma_{ij}\, q_{j*}/[(1 - C_{jj})\, \hat{c}_{ij}]$. Using the identity $q^D_{j*} = (1 - C_{jj})q_{j*}$, this can be written equivalently as $\delta_{ij}\, \gamma_{ij}\, q_{j*}/\hat{c}_{ij}$, expressed relative to the gross output q_{j*}.

(c) **Household final demand**: from (4), each household allocates $\alpha_{ih}\, B_h$ to sector i; the corresponding physical demand is $\alpha_{iH}\, Y^D_*/p_i$ at steady state, where the macroscopic share α_{iH} aggregates over households.

(d) **Government final demand**: from (23), the government allocates $\alpha^G_i\, C^G_*$ to sector i, for a physical demand of $\alpha_{iG}\, C^G_*/p_i$ (we relabel $\alpha^G_i \equiv \alpha_{iG}$ in line with the steady-state notation of Section 5.3).

(e) **Foreign demand**: from (24), the rest of the world allocates $\alpha^R_i\, c^R Y^R$ to domestic sector i; the corresponding physical demand is $\alpha^R_i\, c^R Y^R/p_i$.

Step 3: Domestic share of each demand component. Each of the five demand components above is, in general, partly satisfied by domestic supply and partly by imports from the rest of the world. Denoting by $\Gamma^i_{Rj}, \Gamma^i_{RH}, \Gamma^i_{RG}$ the import shares for the demand of, respectively, sector j (both intermediate input and capital good), households, and government, the *domestic* fraction of each component is $1-\Gamma^i_{Rj}, 1-\Gamma^i_{RH}, 1-\Gamma^i_{RG}$. The foreign demand component (e) is by construction directed entirely at domestic sellers, with the supplier-weight matrix ω^R_{ki} aggregated into the macroscopic foreign expenditure share α^R_i.

Step 4: Physical sales of sector i. Combining Steps 2 and 3, the steady-state physical sales of sector i to all (domestic and foreign) buyers, denoted q^D_{i*}, equal

$$q^D_{i*} \;=\; \sum_j (1-\Gamma^i_{Rj}) \left[C_{ij} q_{j*} + \frac{\delta_{ij}\,\gamma_{ij}}{\hat{c}_{ij}}\, q_{j*} \right] + (1-\Gamma^i_{RH}) \frac{\alpha_{iH} Y^D_*}{p_i} + (1-\Gamma^i_{RG}) \frac{\alpha_{iG} C^G_*}{p_i} + \alpha^R_i \frac{c^R Y^R}{p_i}. \tag{56}$$

The left-hand side is the *net* demand received by sector i (net of own-product use, since the latter does not appear on the intermediate-input side: when $j = i$, the $C_{ii} q_{i*}$ term represents internal consumption, not a market transaction). To obtain a system in gross output, we use the steady-state identity $q^D_{i*} = (1 - C_{ii})\, q_{i*}$ on the left-hand side, transferring the own-product term to the right:

$$q_{i*} \;=\; C_{ii}\, q_{i*} \;+\; \sum_j (1 - \Gamma^i_{Rj}) \left[C_{ij} q_{j*} + \frac{\delta_{ij}\,\gamma_{ij}}{\hat{c}_{ij}}\, q_{j*} \right]_{j \neq i} + \cdots\,, \tag{57}$$

where "$\cdots$" collects the household, government, and foreign demand components, and the bracket retains the same algebraic form as in (56) except that the diagonal term ($j = i$) is now subtracted out and absorbed into the $C_{ii} q_{i*}$ on the right. Equivalently, treating the diagonal as exogenous to the market and including only off-diagonal trade in the Γ^i_R correction, the equation can be written compactly as a system in q_{j*} that includes $j = i$ on the right-hand side, with $\Gamma^i_{Ri} = 0$ by convention.

Step 5: Conversion to nominal form. Multiplying both sides of (57) by p_i and using the definitions $A_{ij} = p_i C_{ij} p_j^{-1}$ and $\Lambda_{ij} = (\delta_{ij}/\hat{c}_{ij})\, p_i \gamma_{ij} p_j^{-1}$ (equations (40)–(41)), each physical quantity q_{j*} is multiplied by p_j inside the sum, and the price ratio p_i/p_j becomes part of A_{ij} and Λ_{ij}:

$$\begin{aligned} p_i C_{ij} q_{j*} \;&=\; A_{ij}\, p_j q_{j*} \;=\; A_{ij}\, x_{j*}, \\ p_i \frac{\delta_{ij}\,\gamma_{ij}}{\hat{c}_{ij}}\, q_{j*} \;&=\; \Lambda_{ij}\, p_j q_{j*} \;=\; \Lambda_{ij}\, x_{j*}, \end{aligned}$$

and similarly $p_i \cdot (\alpha_{iH} Y^D_*/p_i) = \alpha_{iH} Y^D_*$, $p_i \cdot (\alpha_{iG} C^G_*/p_i) = \alpha_{iG} C^G_*$, and $p_i \cdot (\alpha^R_i c^R Y^R/p_i) = \alpha^R_i c^R Y^R$ (since the foreign-demand term is already nominal). Substituting and collecting yields exactly equation (42) of Proposition 3, completing the derivation.

B Existence and uniqueness via Perron–Frobenius

This appendix proves Proposition 4. The strategy is parallel to Appendix A.3 of MAIO1: under Assumption 1, the steady-state system reduces to a non-negative linear system; a Perron–Frobenius-type spectral-radius condition on the augmented system matrix delivers existence, uniqueness, and global stability.

Step 1: Augmented linear system. Equation (42) expresses $\boldsymbol{x}_*$ as a function of the two macroscopic scalars Y^D_* and C^G_*. Equation (46) expresses Y^D_* as a linear function of $\boldsymbol{x}_*$: profits $\Pi^H_* = \sum_i \pi_{i*}$ are linear in $\boldsymbol{x}_*$ through (47); the wage bill W_* is linear in $\boldsymbol{x}_*$ through $W_* = \sum_i w_i C_{wi}\, q_{i*} = \sum_i w_i C_{wi} x_{i*}/p_i$; and the transfer term T_* is linear in the average wage $\bar{w}_*$, hence in $\boldsymbol{x}_*$. Equation (48) expresses $C^G_* = \mathcal{T}_* - T_*$, which is linear in $\boldsymbol{x}_*$ through the tax block.

We collect $(\boldsymbol{x}_*, Y^D_*, C^G_*) \in \mathbb{R}^{N_A+2}$ into the augmented vector $\boldsymbol{v}_* = (x_{1*}, \ldots, x_{N_A *}, Y^D_*, C^G_*)^\top$ and write the steady-state equations in matrix form

$$\boldsymbol{v}_* \;=\; \boldsymbol{T}\, \boldsymbol{v}_* + \boldsymbol{c}, \tag{58}$$

where $\boldsymbol{T} \in \mathbb{R}^{(N_A+2)\times(N_A+2)}$ and the constant vector $\boldsymbol{c}$ collects the (only) exogenous term, the foreign demand $\alpha_i^R c^R Y^R$ in the first N_A rows. The block structure of $\boldsymbol{T}$ is

$$\boldsymbol{T} \;=\; \begin{pmatrix} \widetilde{\boldsymbol{M}} & \boldsymbol{u}_H & \boldsymbol{u}_G \\ \boldsymbol{r}_\Pi^\top + \boldsymbol{r}_W^\top & 0 & 0 \\ \boldsymbol{r}_T^\top & 0 & 0 \end{pmatrix}, \tag{59}$$

where $\widetilde{\boldsymbol{M}}$ is the augmented domestic-coefficient matrix of (43); the column vectors $\boldsymbol{u}_H, \boldsymbol{u}_G \in \mathbb{R}^{N_A}$ collect the household and government share components, $(\boldsymbol{u}_H)_i = (1 - \Gamma^i_{RH})\,\alpha_{iH}$ and $(\boldsymbol{u}_G)_i = (1 - \Gamma^i_{RG})\,\alpha_{iG}$; the row vector $\boldsymbol{r}_\Pi$ encodes the linear dependence of aggregate profits on $\boldsymbol{x}_*$ via (47) (with elements $(\boldsymbol{r}_\Pi)_i = (1-\tau^C)[1-\tau_i^V/(1+\tau_i^V) - \sum_{j\neq i} A_{ji} - w_i C_{wi}/p_i - \sum_j \Lambda_{ji}]^+$); the row vector $\boldsymbol{r}_W$ encodes the wage-bill component, with elements $(\boldsymbol{r}_W)_i = w_i C_{wi}/p_i$; and $\boldsymbol{r}_T$ encodes the tax-net-of-transfer component ($(\boldsymbol{r}_T)_i = \tau_i^V/(1+\tau_i^V) + \tau^C \cdot$[net-earnings coefficient]$_i -$ [transfer share]$_i$).

Step 2: Non-negativity of $\boldsymbol{T}$. Under Assumption 1 (non-negative parameters, no rationing), the matrix $\widetilde{\boldsymbol{M}}$ is non-negative because $\boldsymbol{A}, \boldsymbol{\Lambda} \geq 0$ and $0 \leq \Gamma^i_{Rj} \leq 1$. The vectors $\boldsymbol{u}_H, \boldsymbol{u}_G$ are non-negative because consumption shares and import quotas lie in $[0,1]$. The vector $\boldsymbol{r}_W$ is non-negative because wages and labour intensities are non-negative. The vector $\boldsymbol{r}_\Pi$ is non-negative under the standard profitability condition that, for every sector i, the gross profit margin (one minus the sum of unit costs and effective taxes) is non-negative; if this condition fails, the corresponding row is set to zero by the $(\cdot)^+$ operator in (47), again preserving non-negativity. Finally, $\boldsymbol{r}_T$ is non-negative under the assumption that the government extracts on net (taxes exceed transfers in steady state), an assumption that the implementation respects through the soft liquidity constraint of Section 2.5. Hence $\boldsymbol{T} \geq 0$.

Step 3: Spectral radius and existence.

Lemma 6 (Existence and uniqueness of the fixed point)**.** *Let $\boldsymbol{T} \geq 0$ be a non-negative square matrix with spectral radius $\rho(\boldsymbol{T}) < 1$, and let $\boldsymbol{c} \geq \boldsymbol{0}$. Then the matrix $\boldsymbol{I} - \boldsymbol{T}$ is invertible, $(\boldsymbol{I}-\boldsymbol{T})^{-1} \geq \boldsymbol{0}$, and the equation $\boldsymbol{v} = \boldsymbol{T}\boldsymbol{v} + \boldsymbol{c}$ has the unique non-negative solution $\boldsymbol{v}_* = (\boldsymbol{I}-\boldsymbol{T})^{-1}\boldsymbol{c}$.*

This is a standard consequence of the Perron–Frobenius theorem and the Neumann-series representation $(\boldsymbol{I} - \boldsymbol{T})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{T}^k$, which converges term-by-term to a non-negative limit when $\rho(\boldsymbol{T}) < 1$; see, e.g., [10, Ch. 8].

Step 4: Spectral radius condition. The condition $\rho(\boldsymbol{T}) < 1$ has a transparent economic interpretation: it says that the combined leakages out of the multi-sector demand multiplier are sufficient to dampen the system. The leakages are: imports ($\Gamma^i_{R\bullet}$); household saving (captured by $1 - \alpha_{iH}$ when expressed as a share of Y^D_*, since households consume Y^D_* exactly at the steady state; the genuine saving leakage is in the dynamic system and vanishes at stationarity); government extraction net of transfers ($\boldsymbol{r}_T$); and depreciation that becomes net leakage to the extent that capital goods are imported ($\Gamma^i_{Rj}\,\Lambda_{ij}$).

Step 5: Global stability. The dynamic counterpart of (58) is the linear recursion $\boldsymbol{v}_{t+1} = \boldsymbol{T}\,\boldsymbol{v}_t + \boldsymbol{c}$. The general solution is $\boldsymbol{v}_t = \boldsymbol{T}^t(\boldsymbol{v}_0 - \boldsymbol{v}_*) + \boldsymbol{v}_*$, which converges to $\boldsymbol{v}_*$ for every initial condition $\boldsymbol{v}_0$ because $\rho(\boldsymbol{T}) < 1$ implies $\boldsymbol{T}^t \to \boldsymbol{0}$. Convergence is therefore global. This completes the proof of Proposition 4. □

C Derivation of the group-level fixed point

This appendix derives equation (50) of Section 5.6 and provides the explicit form of the matrix $\mathcal{A}$ and the vector $\boldsymbol{b}$ that characterise the steady-state distribution of disposable income across heterogeneous household groups. The construction parallels Appendix B.3 of [1], with two structural extensions: the wage channel contributes a labour-income component, and the production system of (45) is augmented by the capital-replacement matrix $\mathbf{\Lambda}$ and the import-share structure $(\Gamma^i_{R\bullet})$.

Step 1: Group-level disposable income. Group $g \in \{1, \ldots, G\}$ contains $N_{H,g}$ households sharing identical individual parameters and the same group-level ownership structure $\boldsymbol{S}_g = (S_{gf})_{f=1,\ldots,N_F}$ over firms. The aggregate disposable income of group g at the steady state is the sum of three components:

$$Y^D_{g*} \;=\; \Pi^H_{g*} \;+\; W_{g*} \;+\; T_{g*}, \tag{60}$$

where Π^H_{g*} is dividend income, W_{g*} is wage income, and T_{g*} is government transfers received by the group. Dividend income at the group level reads

$$\Pi^H_{g*} \;=\; \sum_f S_{gf} \sum_k \Omega_{fk}\, \pi_{k*} \;=\; \sum_f S_{gf}\, \pi^F_{f*}, \tag{61}$$

where $\pi^F_{f*} = \sum_k \Omega_{fk}\, \pi_{k*}$ is the firm-level profit aggregated from owned KAUs. Under Assumption 1 (i) (one KAU per sector), π^F_{f*} reduces to the profit of the sector owned by firm f; we map firms to sectors and write $\pi^F_{f*} = \pi_{i(f)*}$.

The wage component is

$$W_{g*} \;=\; \omega^W_g\, W_*, \qquad W_* \;=\; \sum_i w_i\, N_{i*}, \tag{62}$$

where $\omega^W_g \in [0,1]$, with $\sum_g \omega^W_g = 1$, is the group's share of the aggregate wage bill. Under the no-rationing benchmark, $\omega^W_g = N_{H,g}/N_H$ when employment is uniformly distributed across groups (every household is employed at the steady state); more generally, ω^W_g may differ from this benchmark if employment is correlated with group identity. We introduce ω^W_g as a free parameter and discuss the symmetric specification in the remark at the end.

The transfer component is

$$T_{g*} \;=\; \frac{N_{H,g}}{N_H}\, T_*, \qquad T_* \;=\; N_H\, \phi^T\, \bar{w}_*, \tag{63}$$

since transfers are uniform per household (equation (22)). Defining the group population share $n_g \equiv N_{H,g}/N_H$, equation (63) becomes $T_{g*} = n_g\, T_*$.

Step 2: Sectoral profits and the production system. Sector profits at the steady state are given by (47). Define the row vector $\boldsymbol{r}_\Pi$ such that

$$\pi_{i*} \;=\; (\boldsymbol{r}_\Pi)_i\, x_{i*}, \qquad (\boldsymbol{r}_\Pi)_i \;=\; (1-\tau^C)\left[1 - \tfrac{\tau^V_i}{1+\tau^V_i} - \sum_{j\neq i} A_{ji} - \tfrac{w_i C_{wi}}{p_i} - \sum_j \Lambda_{ji}\right]^+, \tag{64}$$

so that aggregate dividend income reads $\sum_i \pi_{i*} = \boldsymbol{r}_\Pi^\top \boldsymbol{x}_*$. Substituting into (61) via the firm-to-sector mapping,

$$\Pi^H_{g*} \;=\; \sum_i \widetilde{S}_{gi}\, \pi_{i*} \;=\; \sum_i \widetilde{S}_{gi}\, (\boldsymbol{r}_\Pi)_i\, x_{i*} \;=\; \tilde{\boldsymbol{S}}_g^\top\, \mathrm{diag}(\boldsymbol{r}_\Pi)\, \boldsymbol{x}_*, \tag{65}$$

where $\widetilde{S}_{gi}$ is the effective ownership of sector i by group g (composition of $\boldsymbol{S}_g$ with the firm-to-sector mapping). Similarly the wage component reads

$$W_{g*} \;=\; \omega^W_g \sum_i w_i\, N_{i*} \;=\; \omega^W_g \sum_i w_i\, C_{wi}\, q_{i*} \;=\; \omega^W_g\, \boldsymbol{w}^\top \boldsymbol{x}_*, \qquad (\boldsymbol{w})_i \;\equiv\; w_i\, C_{wi}/p_i. \tag{66}$$

The transfer component $T_{g*} = n_g\, T_*$ depends on the average wage $\bar{w}_*$, which is itself a linear function of $\boldsymbol{x}_*$ at the steady state; we collect this dependence in a row vector $\boldsymbol{r}^{(g)}_T$, with $T_{g*} = \boldsymbol{r}^{(g)\top}_T\, \boldsymbol{x}_*$.

Step 3: Production system as a function of Y_*^D and group shares. From the Leontief representation (45), sectoral production is

$$\boldsymbol{x}_* \;=\; \widetilde{\boldsymbol{L}}\,\boldsymbol{f}_*, \qquad \boldsymbol{f}_* \;=\; (1-\boldsymbol{\Gamma}_{RH})\,\boldsymbol{\alpha}_{H*}\,Y_*^D + (1-\boldsymbol{\Gamma}_{RG})\,\boldsymbol{\alpha}_G\,C_*^G + \boldsymbol{\alpha}_R\,c^R Y^R, \tag{67}$$

where $\boldsymbol{\Gamma}_{RH}, \boldsymbol{\Gamma}_{RG}$ are diagonal matrices of import shares. The macroscopic household consumption share is

$$\boldsymbol{\alpha}_{H*} \;=\; \sum_g s_{g*}\,\boldsymbol{\alpha}_g, \tag{68}$$

which substituted into (67) makes $\boldsymbol{x}_*$ a linear function of Y_*^D and the group income shares $\boldsymbol{s}_*$:

$$\boldsymbol{x}_* \;=\; \widetilde{\boldsymbol{L}}\,(1-\boldsymbol{\Gamma}_{RH})\left[\sum_g s_{g*}\,\boldsymbol{\alpha}_g\right] Y_*^D \;+\; \boldsymbol{x}_*^{(0)}, \tag{69}$$

where $\boldsymbol{x}_*^{(0)} = \widetilde{\boldsymbol{L}}\,[(1-\boldsymbol{\Gamma}_{RG})\,\boldsymbol{\alpha}_G\,C_*^G + \boldsymbol{\alpha}_R\,c^R Y^R]$ collects the components of $\boldsymbol{x}_*$ that do not depend on $\boldsymbol{s}_*$.

Step 4: Group income shares as a fixed point. Combining (65)–(66), together with $T_{g*} = \boldsymbol{r}_T^{(g)\top}\,\boldsymbol{x}_*$, into the group-level identity (60), we obtain

$$Y_{g*}^D \;=\; \left[\,\tilde{\boldsymbol{S}}_g^\top\,\mathrm{diag}(\boldsymbol{r}_\Pi) + \omega_g^W\,\boldsymbol{w}^\top + \boldsymbol{r}_T^{(g)\top}\,\right]\boldsymbol{x}_* \;\equiv\; \boldsymbol{\eta}_g^\top\,\boldsymbol{x}_*, \tag{70}$$

where the row vector $\boldsymbol{\eta}_g \in \mathbb{R}^{N_A}$ collects all linear contributions of $\boldsymbol{x}_*$ to group g's income.

Substituting (69) into (70) and using $s_{g*} = Y_{g*}^D / Y_*^D$, we obtain

$$s_{g*}\,Y_*^D \;=\; \boldsymbol{\eta}_g^\top\,\widetilde{\boldsymbol{L}}\,(1-\boldsymbol{\Gamma}_{RH})\left[\sum_{g'} s_{g'*}\,\boldsymbol{\alpha}_{g'}\right] Y_*^D \;+\; \boldsymbol{\eta}_g^\top\,\boldsymbol{x}_*^{(0)}. \tag{71}$$

Dividing through by Y_*^D and using $\boldsymbol{x}_*^{(0)}/Y_*^D = \boldsymbol{\xi}^{(0)}$ (a vector independent of $\boldsymbol{s}_*$ once C_*^G, Y^R are pinned down by the closure conditions), equation (71) becomes

$$s_{g*} \;=\; \sum_{g'} \mathcal{A}_{gg'}\,s_{g'*} \;+\; b_g, \tag{72}$$

with the explicit coefficients

$$\mathcal{A}_{gg'} \;=\; \boldsymbol{\eta}_g^\top\,\widetilde{\boldsymbol{L}}\,(1-\boldsymbol{\Gamma}_{RH})\,\boldsymbol{\alpha}_{g'}, \qquad b_g \;=\; \boldsymbol{\eta}_g^\top\,\boldsymbol{\xi}^{(0)}. \tag{73}$$

Equation (72) is the explicit form of equation (50). The constraint $\sum_g s_{g*} = 1$ removes one degree of freedom in the solution and makes the system consistent.

Step 5: Existence and the structural invariance of the production system. The fixed-point equation (72) admits a non-negative solution $\boldsymbol{s}_*$ whenever the matrix $\mathcal{A}$ satisfies $\rho(\mathcal{A}) < 1$, by the same Perron–Frobenius argument of Appendix B. Once $\boldsymbol{s}_*$ is determined, the macroscopic propensity $\boldsymbol{\alpha}_{H*}$ is given by (68), and the production system (42) retains its form with $\boldsymbol{\alpha}_{H*}$ in place of $\boldsymbol{\alpha}_H$. The structural invariance discussed in the remark at the end of Section 5.6 is a direct consequence: heterogeneity enters the steady state only through the weighted average $\boldsymbol{\alpha}_{H*} = \sum_g s_{g*}\,\boldsymbol{\alpha}_g$, and analogously through λ_{H*}^T for the wealth distribution. □

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Table 1: Sequence of events within period t. "KAU" denotes a local KAU; "HH" a household; "Firm" a non-financial firm; "Bank" the single commercial bank; "Gov" the government; "RoW" the rest of the world. Cross-references to equations point to the sections in which the corresponding rule is formalised.

#	**Phase**	**Agents**	**Key actions**
1	Forecasting	KAUs	Form demand expectation $q^{De}_{k,t} = q^{D}_{k,t-1}$; set stock and capital targets.
2	Production planning	KAUs	Plan production $\hat{q}_{k,t}$, intermediate-input demand, capital demand, and labour demand (Section 2.3).
3	Liquidity assessment	KAUs → Firms	KAUs compute $\Lambda^{\mathrm{op}}_{k,t}$ and report to the firm; firms aggregate to the funding gap $L^{d}_{f,t}$ (eq. (34)).
4	Credit market	Firms ↔ Bank	Firms submit $L^{d}_{f,t}$; bank applies the granting rule (17); granted loans are allocated within each firm according to (35).
5	Labour market	Unemployed HHs ↔ KAUs	KAUs post vacancies or fire (eq. (29)); unemployed households are matched at random to open vacancies (Section 3.2).
6	Loan repayments and wage payments	Firms → Bank, KAUs → HHs	Firms service outstanding loans (eqs. (18)–(36)); wages are paid (eq. (30)); firm dividends are distributed to households.
7	Bank dividends	Bank → HHs	The bank's interest income is distributed equally across all households (eq. (19)).
8	Production and pricing	KAUs	Output is realised (eq. (32)); prices are posted (eq. (15)).
9	Budget formation	HHs, Gov, RoW, KAUs	Households compute income, taxes (accrual basis), disposable income, and consumption budget; the government distributes unemployment benefits and transfers, then sets its public-consumption budget; the rest of the world sets its import budget; KAUs scale demanded quantities to current cash.
10	Capital depreciation and commitment	KAUs	Capital stocks are depreciated; own-product capital is committed for the next period (Section 2.3).
11	Goods market	HHs, KAUs, Gov, RoW (buyers); KAUs, RoW (sellers)	R rounds of decentralised matching per commodity (Section 3.1); transactions clear and stocks update.
12	Period closure	KAUs, Firms, Bank, HHs, Gov	KAUs compute earnings and revalue input stocks; firms and the bank age their loan portfolios; households and KAUs pay accrued taxes; the government closes the period account.