PRICE SETTING GENERAL EQUILIBRIUM WITH INTERMEDIATE GOODS

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Abstract: Purely based on normal assumptions on consumer preferences and production technology, the literature has not established the existence of any equilibrium with imperfect competition beyond that of Negishi equilibrium. In a general equilibrium model with intermediate goods, one firm's demand function cannot be determined unless we know its downstream firms' demand functions. Hence even the existence of demand functions becomes problematic. We introduce generalized first-order equilibrium (GFE), where every firm perceives a linear demand curve, whose slope is bounded by the left and right limits of the objective demand derivatives. Given normal consumer preferences and technology, there always exists a GFE in a price setting economy with intermediate goods.

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1. Introduction

While the existence of a competitive general equilibrium has been established purely based on assumptions on consumer preference and technology, its counterpart with imperfect competition has to rely on ad hoc and unjustifiable assumptions (see Bonnano [1990]). For instance, the profit function is often assumed to be quasi-concave (Arrow and Hahn [1971], Gabszewiz and Vial [1972], Fitzroy [1974], Nikaido [1975], Laffont and Laroque [1976], Benassy [1988]). Otherwise, as shown by Roberts and Sonnenschein (1977), standard assumptions on preference and technology cannot ensure the existence of a general equilibrium with imperfect competition, in the sense of a genuine Nash equilibrium. Later Bonnano (1988) showed that in a partial equilibrium duopoly model, even a local maximum equilibrium may fail to exist. Therefore, to obtain an existence result of some sort of general equilibrium with imperfect competition, one has to accept bounded rationality below the level of local profit maximization.

Negishi (1961) first explored along this road and proved the existence of Cournot-Walras general equilibrium where every firm's first-order condition for profit maximization holds subject to a perceived linear demand curve that predict the level of demand correctly in equilibrium. The weakness of his concept is lack of restrictions on the slopes of subjective demand curves, so that almost any output/price decision could be thought as "optimal". A natural improvement is to require each firm estimate its slope correctly, so that its first-order condition for profit maximization actually holds in equilibrium. We call it "first-order equilibrium". Silvestre (1977) and Gary-Bobo (1989) prove the existence of this type of general equilibrium with price and quantity competition respectively. Their results rely on two assumptions: (i) the specific production technology or total absence of intermediate goods; (ii) downward sloping and differentiable demand curves.

In spite of their crucial importance, intermediate goods have not been incorporated into imperfectly competitive general equilibrium models satisfactorily. In Gabszewicz and Vial (1972), Marshak and Selton (1974) and Gary-Bobo (1989), all inputs come from consumers' endowments. Negishi (1961) and Fitzroy (1974) do not allow imperfectly competitive firms sell products to each other. Nikaido (1975) and Silvestre (1977) assume that all goods, including intermediate ones, to be produced by Leontief technology only, so that a firm's input demand is proportional to its output, and the whole demand vector can be solved through an inverse matrix. Benassy (1988) incorporates intermediate goods with convex technology, which is more general and can approximate Leontief production functions. He assumes, however, downstream firms do not adjust their output prices to any input price change, which is not a rational behavior.

The difficulty to incorporate intermediate goods arises when firms mutually demand for each other's products. For example, a computer producer may need software and sell its product to the software producer as well. Its demand function depends on how the software firm adjusts its input demand to the computer price. While making such adjustment, the software firm should take into account of the reaction of the computer firm to the software price change. But the computer firm's reaction to its input price depends on its output market. We are back to square one. In sun a complex network or circles, we may not be able to find any firm's demand function unless we get all firms' demand functions.

To overcome this difficulty, it seems necessary to limit firms' demand knowledge to a liner function, which matches the true demand in its quantity level and the slope estimation. Hence intermediate goods actually increase the appeal of first-order equilibrium. However, as we mentioned earlier, first-order equilibrium relies on the assumption of demand differentiability, and introducing intermediate goods actually intensifies the problem of this assumption. In an exchange economy, the differentiability can be justified by numerous small consumers so that the demand curve becomes smooth by aggregation[1]. But this argument is less convincing for firms' input demand functions.

Unfortunately, the assumption of demand differentiability is essential to first-order equilibrium. In a partial equilibrium model, Bonanno and Zeeman (1985) show the existence of first-order equilibrium without assuming downward sloping demand curves but the demand differentiability is still assumed. In fact, without demand differentiability, first-order equilibrium may fail to exist. Bonanno (1988) provides a duopoly example consistent with standard utility maximization, which does not possess either Nash or local maximum equilibrium. At the only possible location for first-order equilibrium, one firm's demand curve is not differentiable[2]. Hence no first-

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[1] Caplin and Nalebuff (1991) obtain an Bertrand equilibrium with strong conditions on the distribution of consumers and their preferences.

[2] Bonanno assumes $D_1 = -0.0014p_1^3 + 0.0748p_1^2 - 1.4796p_1 + p_2$, $D_2 = 10.5 + 1.9p_1 - 2p_2$ for $p_2 \leq 0.94p_1 + 1/3$, but $D_2 = 10 + 0.49p_1 - 0.5p_2$ otherwise. The only possible first-order equilibrium occurs at the intersection of firm 1's "reaction curve", which satisfies its first-order condition and firm 2's "reaction curve", which coincides with the line $p_2 = 0.94p_1 + 1/3$, i.e., (12.7,12.3). Given Bonanno's assumption, firm 2's demand function is not differentiable at this point.

order equilibrium exists. Moreover, we can show in a simple duopoly case that first-order equilibrium may not exist even though Nash equilibrium does[3].

One possible solution to restore firms' first-order conditions is to assign certain slope estimations at non-differentiable points. It is easy to verify that in both of Bonanno's and our examples, we can find particular slope estimations between the limits of demand derivatives such that firms' first-order profit maximization condition hold[4]. In each of these cases, we can say that the slope estimation is reasonable, because it lies within two possible "correct" values.

To establish the existence of some sort of equilibrium with intermediate goods and without any ad hoc assumptions on demand curves, we introduce "a generalized first-order equilibrium" (GFE), which is identical to a first-order equilibrium when demand is differentiable; otherwise, the slope of subjective demand must be bounded by the left and right limits of the derivatives of the objective demand. Purely based on assumptions regarding preference and technology, we will prove the existence of such a GFE in a price setting economy with intermediate goods.

The next section of the paper introduces the model. Section three outlines the proof. Section four proves the existence and section five discusses its properties, which is followed by concluding remarks in the final section.

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[3] Let $U = m + x_1 + x_2 - 0.5(x_1^2 + x_2^2 + x_1x_2)$, where m is money. Demand: $x_i = 2(1 - 2p_i + p_j)/3$ for $x_1, x_2 > 0$, $x_i = 1 - p_i$ if $1 - 2p_i + p_j > 0$ and $1 - 2p_j + p_i \leq 0$. Cost: $c_1 = 0.25$, $c_2 = 0.8$. The Nash equilibrium: $p_1 = 0.6$, $p_2 = 0.8$, where the demand is not differentiable. No first-order equilibrium exists.

[4] In Bonanno's example, point $p_1 = 12.7$, $p_2 = 12.3$ satisfies firm 2's first-order condition if its slope estimation is -0.8217, which is between the two limits of derivatives of its true demand curve, –2 and –0.5. In our example, the Nash equilibrium $p_1 = 0.6$, $p_2 = 0.8$, satisfies firm 1's first-order condition if its slope estimation is -8/7, between the two limits of its demand curve derivatives, -4/3 and -1.

2. Model

The economy consists of m price taking consumers, denoted by i = 1, 2, . . . m, and n price setting firms, denoted by k = 1, 2. . n. Every firm k produces a single product. It employs h_k amount of labor and purchases an input vector $\mathbf{z}_k$. Given h_k and $\mathbf{z}_k$, firm k's output f_k is determined by a production function $f_k(\mathbf{z}_k,h_k)$. A firm would not demand all goods in R^n_+, in particular not its own product. We assume firm k only chooses $\mathbf{z}_k$ from its input space $R_k \subseteq R^{n-1}_+$.

Let x_{ik} be consumer i's demand for firm k's product and let z_{jk} ($j \neq k$) be firm j's input demand for firm k's product. The total demand for firm k's product is thus $D_k = \sum_{i=1}^{m} x_{ik} + \sum_{j=1}^{n} z_{jk}$. Let w be the wage rate and $\mathbf{p}_{-k}$ (which does not contain p_k) be the price vector for $\mathbf{z}_k$. Firm k's profit $\pi_k = p_k \min(f_k,D_k) - \mathbf{p}_{-k}\cdot\mathbf{z}_k - wh_k$. Consumer i owns a fraction θ_{ik} of firm k ($0 \leq \theta_{ik} \leq 1$). All firms are privately owned, i.e., $\sum_{i=1}^{m} \theta_{ik} = 1$ for every k.

Given a fixed time endowment L_i, every consumer i chooses a labor supply $q_i \leq L_i$, enjoying a leisure time $\ell_i = L_i - q_i$, and expecting a salary wq_i. In addition she receives a dividend $d_i = \sum_{k=1}^{n} \theta_{ik}\pi_k$, and her total income is $wq_i + d_i$. She purchases a consumption bundle $\mathbf{x}_i$, and obtains a utility $u_i(\mathbf{x}_i,\ell_i)$. We assume she chooses $\mathbf{x}_i$ from her consumption space $R_i \subseteq R^n_+$. Her labor supply, q_i, can be negative, i.e., she may purchase domestic labor service to expand her leisure time beyond L_i. Examples include babysitting, cleaning, escorting etc. There is no satiation point, and no saving, so $d_i + wq_i = \mathbf{p}_i \cdot \mathbf{x}_i$, where $\mathbf{p}_i$ is the price vector for $\mathbf{x}_i$.

Every buyer, a consumer or firm, is non-strategic, i.e., she does not expect her purchase affects any price or her dividend. No firm appreciates the "Ford effect", the impact of its price/sale on consumers' income, and every firm perceives a linear demand curve for its own product.

Now we describe our economy based on the primary data, i.e., firms' production functions and consumers' utility functions.

Assumption 1: Every $f_k(\mathbf{z}_k,h_k)$ has continuous and bounded first- and second-order derivatives, is non-decreasing and strictly concave. The determinants of its Hessian matrix and bordered Hessian matrix are bounded away from zero.

Assumption 2: Every $u_i(\mathbf{x}_i,\ell_i)$ has continuous and bounded first- and second-order derivatives, is non-decreasing and strictly quasi-concave. The determinant of its bordered Hessian matrix is bounded away from zero.

It is known that the existence of a general equilibrium with imperfect competition can be disturbed by arbitrary choices of price normalization (Dierker and Grodal [1986] and [1999], Boehm [1994]). Instead of leaving any price normalization possible as in a competitive economy, Nikaido (1975) and Silvestre (1977) use labor as the numeraire to normalize the wage rate to 1, giving labor a special status as classic economists do. We follow Nikaido (1975) and Silvestre (1977) and further assign labor some unique properties in the economy.

Assumption 3: (i) Labour is essential in production, i.e., $f_k(\mathbf{z}_k,0) = 0$ for every k; (ii) The marginal product of labour, $\partial f_k/\partial h_k$, is bounded above zero; (iii) For every consumer i, her marginal utility of leisure, $\partial u_i/\partial \ell_i$, is bounded above zero.

To justify part (i), we notice that virtually all firms use labor as input (we count self-employment as labor). Part (ii) implies that no other good is essential. Part (iii) can be partially justified by part (ii), i.e., labor can be potentially used to meet any possible demand. Since the marginal utility of leisure is bounded above zero, so must be the equilibrium wage rate. Thus we can use the wage rate as the price denominator. From now on prices are relative to the wage rate, and profits are measured in terms of units of labor.

We give our definition of a generalized first-order general equilibrium in a price setting economy with intermediate goods, in terms of prices, consumption, labor supply, input demand, labour demand, and firms' slope estimations.

Definition 1: We say a point ($\mathbf{p}$*,$\mathbf{x}$*,$\mathbf{z}$*,$\mathbf{h}$*,$\mathbf{q}$*,$\mathbf{s}$*) is a generalized first-order general equilibrium if it satisfies the following conditions:

(I) Every consumption bundle $\mathbf{x}_i^*$ and labor supply q_i^* maximize the utility $u_i(\mathbf{x}_i,L_i\text{-}q_i)$ given $\mathbf{p}_i^*$, and $d_i^* = \sum_{k=1}^{n}\theta_{ik}\,[p_k^*\,f_k(\mathbf{z}_k^*,h_k^*) - \mathbf{p}_{-k}^*\cdot\mathbf{z}_k^* - h_k^*]$, i.e.,

$$u_i(\mathbf{x}_i^*,L_i - q_i^*) \geq u_i(\mathbf{x}_i,L_i - q_i) \text{ for any } (\mathbf{x}_i,q_i) \text{ such that } \mathbf{p}^*\cdot\mathbf{x}_i - q_i \leq d_i^*.$$

(II) (i) Given $\mathbf{p}_{-k}^*$ for any firm k, if $D_k^* = \sum_{i=1}^{m} x_{ik}^* + \sum_{j=1}^{n} z_{jk}^* > 0$, we have $(p_k^*,\mathbf{z}_k^*,h_k^*)$ maximizes profit, $p_k\min.[f_k(\mathbf{z}_k,h_k),D_k^* + s_k^*(p_k - p_k^*)] - \mathbf{p}_{-k}^*\cdot\mathbf{z}_k - h_k$, i.e.,

$$p^*_k D^*_k - \mathbf{p}^*_{-k} \cdot \mathbf{z}^*_k - h^*_k \geq p_k \min[D^*_k + s^*_k(p_k - p^*_k), f_k(\mathbf{z}_k, h_k)] - \mathbf{p}^*_{-k} \cdot \mathbf{z}_k - h_k$$

where $s^*_k \in [\partial D^*_k/\partial p_k|_{+0}, \partial D^*_k/\partial p_k|_{-0}]$;

(ii) If $D^*_k = 0$, then $p^*_k f_k(\mathbf{z}_k,h_k) - \mathbf{p}_{-k} \cdot \mathbf{z}_k - h_k \leq 0$, and $D_k = 0$ for any $p_k \geq p^*_k$.

(III) The labor market clears, i.e., $\sum_{i=1}^{m} q^*_i = \sum_{k=1}^{n} h^*_k$.

Condition (II.ii) rules out a trivial equilibrium, where some firm sets a price too high to have any positive demand, while a positive profit can be made if it lowers its price sufficiently, but not by a small margin. In evaluating $\partial D^*_k/\partial p_k$, we require that each $\partial x^*_{ik}/\partial p_k$ to be consistent with (I) for each consumer i, and $z^*_{jk}/\partial p_k$ to be consistent with (II) for firm j. Thus, as firm k perceives a linear demand curve, its slope estimation is justified by its downstream firms' responses, which depend on their linear demand perceptions for their products, which are justified by their downstream firms' reactions and so forth. In equilibrium every firm maximizes its profit subject to its slope estimation, which is consistent with other firms' profit maximization subject to their slope estimations and so forth.

3. Proof outline

Our existence proof of GFE follows several steps, each of them is explained here and the formal proof will be presented in the next section.

(i) ***Consumer demand***: We show that any consumer i's demand for firm k's product x_{ik} is a continuous function. It is differentiable except for finite "switching points". The same property holds for the aggregate consumer demand function.

(ii) ***Input demand***: We introduce an internal price c_k for any firm k. With such a c_k, the profit maximization could be viewed as being delegated to two managers, in charge of production and price setting respectively. The production manager chooses input and labor demand to maximize profit, taking c_k as the output price.

(iii) ***Bounded demand***: Combining all firms' input demand with the aggregate consumer demand, we get the total demand for firm k's product. We will show that every firm k's demand, its price and internal price are all bounded.

(iv) ***Price setting***: We then turn to firms' price setting by marketing managers, who perceives a linear demand curve and chooses a price p_k to maximize profit taking the internal price c_k as the marginal cost. We will show such a price can be defined as an upper hemi-continuous correspondence. Given this price, the expected demand level will be determined.

(v) ***Internal price***: To match firm k's marketing manager's expected demand with the production manager's output decision, we need to adjust the internal price, c_k. From this requirement, c_k can be determined as a continuous function.

(vi) ***Slope of demand curve***: In evaluating the slope of firm k's demand curve, we allow its input demanders adjust not only their inputs, but also their outputs and hence their output prices, which depend on the slope estimations of their own demand curves. We show the slope of firm k's demand curve is always bounded and upper-hemi-continuous.

(vii) ***Fixed point***: We will show that, all firms' demands, prices, internal prices and slope estimations form an upper hemi-continuous correspondence from a compact set to itself, and hence have a fixed point.

(ix) ***Equilibrium existence***: We will show the fixed point satisfies all conditions (I) – (III) in our equilibrium definition, hence a GFE exists in our price setting economy with intermediate goods.

4. Existence proof

In this section we will prove the following:

Theorem: *Given our Assumptions 1 - 3, there is a generalized first-order equilibrium satisfying conditions (I) – (III) of Definition 1.*

(i) ***Consumer demand***: We start with consumer i's choice of q_i and $\mathbf{x}_i$.

Definition 2: Given $\mathbf{p}_i$ and d_i, q_i and $\mathbf{x}_i$ maximize the utility function $u_i(\mathbf{x}_i,L_i\text{-}q_i)$ subject to $\mathbf{p}_i\cdot\mathbf{x}_i = d_i + q_i$, and are written as

$$q_i = q_i(\mathbf{p}_i,d_i) \tag{1}$$

$$x_{ik} = x_{ik}(\mathbf{p}_i,d_i) \tag{2}$$

The solution must be unique since $u_i(\mathbf{x}_i,\ell_i)$ is strictly quasi-concave. If there were two equally desirable consumption and leisure choices, any linear combination of them would yield a strictly higher utility and still satisfy the budget constraint. This leads to a contradiction. As $u_i(\mathbf{x}_i,\ell_i)$ is continuous, $(\mathbf{x}_i,q_i)$ is continuous in $(\mathbf{p}_i,d_i)$.

Suppose, for a moment, that none of the product demanded by the consumer switches between 0 and a positive amount. Then, since the bordered Hessian of $u_i(\mathbf{x}_i,q_i)$ is non-singular, (2) is differentiable, in particular, $\partial x_{ik}/\partial p_k$ exits when $x_{ik} > 0$ (see Katzner 1968). Moreover, as the bordered Hessian has its elements all bounded, and its determinant bounded away from zero, $\partial x_{ik}/\partial p_k$ must be bounded.

However, if consumer i's demand for any product, not necessarily good k, switches around zero, the demand function for good k may not be differentiable. For instance, in our earlier duopoly example, when the representative consumer's demand for one firm's product drops to zero, both firms' demand functions become non-differentiable. Nevertheless, the left and right limits of the derivative do exist, corresponding to the situations before the switch and afterwards. Therefore, we still have $\partial x_{ik}/\partial p_k|_{+0}$ and $\partial x_{ik}/\partial p_k|_{-0}$, and they are bounded.

Let $\mathbf{d}$ be the m×1 vector of consumer dividends. As every x_{ik} is continuous in $(\mathbf{p}_i,d_i)$, the aggregate consumer demand for firm k's product, $\sum_{i=1}^{m} x_{ik} \equiv X_k$, must be continuous in $(\mathbf{p},\mathbf{d})$. Furthermore, if no consumer switches her consumption on any good around zero, X_k is differentiable. As for a single consumer, at all switching points, the derivative of the aggregate consumer demand function still has the left and right limits, $\partial X_k/\partial p_k|_{+0}$ and $\partial X_k/\partial p_k|_{-0}$, and they are bounded. These switching points must be finite given finite consumers and goods.

(ii) ***Input demand***: To consider firm k's input choice, we let its production manager choose $(\mathbf{z}_k,h_k)$ to maximize the production profit taking c_k as the output price.

Definition 3: Given c_k and $\mathbf{p}_{-k}$, firm k's input $(\mathbf{z}_k,h_k)$ maximizes profit $c_k f_k(\mathbf{z}_k,h_k) - \mathbf{p}_{-k} \cdot \mathbf{z}_k - h_k$, and are written as:

$$h_k = h_k(\mathbf{p}_{-k},c_k) \tag{3}$$

$$z_{kj} = z_{kj}(\mathbf{p}_{-k},c_k) \qquad (j \neq k) \tag{4}$$

As $f_k(\mathbf{z}_k,h_k)$ is strictly concave, the solution of $(\mathbf{z}_k,h_k)$ is unique. Otherwise, any convex combination of two "optimal" points would give a higher profit. The unique solution must be a continuous function of the internal and input prices.

Conversely to (4), any firm j's demand for firm k's product, z_{jk}, can be written as a function of $(\mathbf{p}_{-j},c_j)$. Let $\mathbf{c}_{-k}$ be an $(n - 1)\times 1$ vector without c_k. The total input demand for firm k's product, $\sum_{j=1}^{n} z_{jk} \equiv Z_k$, must be a continuous function of $(\mathbf{p},\mathbf{c}_{-k})$.

(iii) ***Bounded demand***: As we showed earlier that X_k is continuous in $(\mathbf{p},\mathbf{d})$ and Z_k is continuous in $(\mathbf{p},\mathbf{c}_{-k})$, the aggregate demand for firm k's product, $\sum_{i=1}^{m} x_{ik} + \sum_{j=1}^{n} z_{jk}$ must be a continuous function of $(\mathbf{p},\mathbf{d},\mathbf{c}_{-k})$, and can be written as $D_k(\mathbf{p},\mathbf{d},\mathbf{c}_{-k})$. To show that any firm k's demand is bounded, we first need to prove its price and internal price are bounded.

Lemma 1: For any k, there exist positive P_k and C_k such that if $p_k \geq P_k$, $D_k(\mathbf{p},\mathbf{d},\mathbf{c}_{-k}) = 0$; if $c_k \leq C_k$, $f_k(\mathbf{z}_k,h_k) = 0$ (see Appendix A for the proof).

From now on we define every internal price c_k and a firm's price p_k in a closed and bounded interval $[C_k,P_k]$.

Then, we let $c_k = P_k$, and $p_j = C_j$ for all $j \neq k$, and solve firm k's input and labor demand from (3) and (4). Denote the solution $h_k(\mathbf{C}_{-k},P_k)$ by $\bar{h}_k$, and $z_{kj}(\mathbf{C}_{-k},P_k)$ by $\bar{z}_{kj}$. Substituting them into the production function, we have $f_k(\bar{z}_k,\bar{h}_k) \equiv \bar{f}_k$. This is firm k's output given the highest possible output price and the lowest possible input prices. Since firm k will never produce more than this level of output in equilibrium, we can confine firm's demand in a closed and bounded interval $[0, \bar{f}_k]$, and write it as:

$$\mathcal{D}_k(\mathbf{p},\mathbf{d},\mathbf{c}_{-k}) = \min.[D_k(\mathbf{p},\mathbf{d},\mathbf{c}_{-k}), \overline{f}_k] \quad (5)$$

(iv) ***Price setting***: Given $\mathcal{D}_k$, p_k and a slope estimation s_k, the marketing manager perceives a linear demand function:

$$Y_k(\mathcal{D}_k,p_k,s_k,\not{p}_k) = \max.[0, \mathcal{D}_k + s_k(\not{p}_k - p_k)] \quad (6)$$

Taking c_k as the constant marginal cost, the marketing manager then sets a price $\not{p}_k$ to maximize the marketing profit.

Definition 4: Firm k's price $\not{p}_k \in [C_k,P_k]$ is chosen to maximize $Y_k(\not{p}_k - c_k)$. We write this mapping from $(\mathcal{D}_k,p_k,c_k,s_k)$ into interval $[C_k,P_k]$ as:

$$\not{p}_k\text{: } \phi_k(\mathcal{D}_k,p_k,c_k,s_k) \quad (7)$$

Lemma 2: (7) is an upper hemi-continuous correspondence (see Appendix B).

Substituting $\not{p}_k$ back to (6), we get firm k's perceived demand Y_k as a function of $(\mathcal{D}_k,p_k,c_k,s_k)$. Denote it by $\mathcal{Y}_k(\mathcal{D}_k,p_k,c_k,s_k)$. As (7) gives a unique value for $\not{p}_k$ except for the case $s_k = 0$, $\mathcal{Y}_k$ is a continuous function in its all arguments.

(v) ***Internal price***: To equalize firm k's expected demand, $\mathcal{Y}_k(\mathcal{D}_k,p_k,c_k,s_k)$, with the output chosen by its production manager, we need to adjust the internal price c_k. As (3) and (4) are continuous in $\mathbf{p}_{-k}$ and c_k, we substitute $h_k(\mathbf{p}_{-k},c_k)$ and $\mathbf{z}_k(\mathbf{p}_{-k},c_k)$ into $f_k(\mathbf{z}_k,h_k)$, and obtain a continuous function $f_k[\mathbf{z}_k(\mathbf{p}_{-k},c_k),h_k(\mathbf{p}_{-k},c_k)] = \mathscr{f}_k(\mathbf{p}_{-k},c_k)$.

Definition 5: We choose $\mathscr{c}_k \in [C_k,P_k]$ to make $\mathcal{Y}_k(\mathcal{D}_k,p_k,\mathscr{c}_k,s_k) = \mathscr{f}_k(\mathbf{p}_{-k},\mathscr{c}_k)$; if $\mathcal{Y}_k = \mathscr{f}_k = 0$, we choose $\mathscr{c}_k$ at the switching point between $\mathscr{f}_k = 0$ and $\mathscr{f}_k > 0$; if $\mathcal{Y}_k > \mathscr{f}_k$ when $\mathscr{c}_k = P_k$, we let $\mathscr{c}_k = P_k$. We write this mapping from $(\mathcal{D}_k,\mathbf{p},s_k)$ to interval $[C_k,P_k]$ as

$$\mathcal{c}_k = \chi_k(\mathcal{D}_k,\mathbf{p},s_k) \tag{8}$$

Lemma 3: (8) is a continuous function (see Appendix C).

(vi) ***Slope of demand curve***: At this moment, we can say nothing about the differentiability of $\mathcal{D}_k(\mathbf{p},\mathbf{d},\mathbf{c}_{-k})$, although we know its component of X_k is differentiable except for switching points. As Z_k is concerned, we need to consider each firm j's input demand response to firm k's price, $\partial z_{jk}/\partial p_k$. If we keep c_j constant, as the bordered Hessian of $f_j(\mathbf{z}_j,h_j)$ is non-singular, by the implicit function theorem, we know $\partial z_{jk}/\partial p_k$ exists and is bounded except for switching points. Unfortunately, such a $\partial z_{jk}/\partial p_k$ would not be optimal as c_j is kept fixed. Instead, firm j should adjust $\mathbf{z}_j$, h_j and p_j (through c_j) together so that its new marginal cost matches its new marginal revenue and its new output matches its new subjective demand. We need to find the value of $\partial z_{jk}/\partial p_k$ taking into account such optimal adjustment of both inputs and outputs.

Lemma 4: If $z_{jk} > 0$, and no switch occurs, there exists $\partial z_{jk}/\partial p_k$, where $(\mathbf{z}_j,h_j)$ satisfies (3) and (4) and c_j satisfies (8). It is always negative, bounded, and continuous in all its arguments (see Appendix D).

When no switch occurs, we can sum $\partial z_{jk}/\partial p_k$'s for all j and $\partial x_{ik}/\partial p_k$'s for all i to get $\partial\mathcal{D}_k/\partial p_k$. It must be non-positive, continuous and bounded. When switches occur, the slope does not exist, but its left and right limits, $\partial\mathcal{D}_k/\partial p_k|_{+0}$ and $\partial\mathcal{D}_k/\partial p_k|_{-0}$, do and must be bounded. Let $\underline{S}_k$ and $\overline{S}_k$ be the lower and upper bounds of $\partial\mathcal{D}_k/\partial p_k|_{-0}$ and $\partial\mathcal{D}_k/\partial p_k|_{+0}$. As $\mathcal{D}_k$ in (5) is a function of $(\mathbf{p},\mathbf{d},\mathbf{c}_{-k})$, and $\mathcal{c}_j$ in (8) is a function of $(\mathcal{D}_j,\mathbf{p},s_j)$, the derivatives depend on $(\mathcal{D}_{-k},\mathbf{p},\mathbf{d},\mathbf{c}_{-k},\mathbf{s}_{-k})$.

Moreover, we refine our definition of firm k's slope estimation according to $\mathcal{D}_k = 0$ or $\mathcal{D}_k > 0$, also whether the switching price is higher, equal or lower than the internal cost c_k. Thus we have a mapping from $(\boldsymbol{\mathcal{D}},\mathbf{p},\mathbf{d},\mathbf{c},\mathbf{s}_{-k})$ to interval $[\underline{S}_k, \overline{S}_k]$.

Definition 6: Given $(\boldsymbol{\mathcal{D}},\mathbf{p},\mathbf{d},\mathbf{c},\mathbf{s}_{-k})$, (i) If $\mathcal{D}_k > 0$, then $s_k \in [\partial\mathcal{D}_k/\partial p_k|_{+0}, \partial\mathcal{D}_k/\partial p_k|_{-0}]$; (ii) If $\mathcal{D}_k = 0$, (ii.a) if the switching price is higher than c_k, $s_k = \partial\mathcal{D}_k/\partial p_k|_{-0}$ evaluated at the switching point; (ii.b) if the switching price = c_k, $s_k \in [0, \partial\mathcal{D}_k/\partial p_k|_{-0}]$; (ii.c) if no switching point exists or it is below c_k, $s_k = 0$. We write this correspondence as:

$$s_k: \sigma_k(\boldsymbol{\mathcal{D}},\mathbf{p},\mathbf{d},\mathbf{c},\mathbf{s}_{-k}) \tag{9}$$

Lemma 5: Correspondence (9) is upper hemi-continuous and gives a negative value bounded away from zero in case (ii.a) of Definition 6 (see Appendix E).

(viii) ***Fixed point***: Now we will establish a fixed point. Before applying the fixed-point theorem, we can simplify some of our earlier expressions. Equations (5) and (9) contain the vector of consumer dividends **d**. Consumer i receives a dividend $d_i = \sum_{k=1}^{n} \theta_{ik}\pi_k$, where $\pi_k = p_k f_k(\mathbf{z}_k, h_k) - \mathbf{p}_{-k}\cdot\mathbf{z}_k - h_k$. Using (3) and (4) we can write π_k as a functions of **p** and c_k, and d_i as a continuous function $d_i = d_i(\mathbf{p},\mathbf{c})$. Substituting d_i into (5) and (9), we write demand function (5) as a new function $\tau_k(\mathbf{p},\mathbf{c})$, and slope correspondence (9) as $\omega_k(\boldsymbol{\mathcal{D}},\mathbf{p},\mathbf{c},\mathbf{s}_{-k})$. Finally, we obtain the following upper hemi-continuous correspondences:

$$\mathcal{D}_k = \tau_k(\mathbf{p},\mathbf{c}) \tag{5'}$$

$$p_k: \phi_k(\mathcal{D}_k, p_k, c_k, s_k) \tag{7}$$

$$c_k = \chi_k(\mathcal{D}_k, \mathbf{p}, s_k) \quad (8)$$

$$s_k: \omega_k(\mathcal{D}, \mathbf{p}, \mathbf{c}, \mathbf{s}_{-k}) \quad (9')$$

(5'), (7), (8), (9') together form an upper hemi-continuous correspondence mapping from a compact and convex set K ≡ $[(\mathcal{D},\mathbf{p},\mathbf{c},\mathbf{s}) \in R^{4n} | 0 \le \mathcal{D}_k \le \overline{f}_k, C_k \le p_k \le P_k, C_k \le c_k \le P_k, \underline{S}_k \le s_k \le \overline{S}_k$, for all k} into itself. By Kakutani's fixed-point theorem, there exists a point $(\mathcal{D}^*,\mathbf{p}^*,\mathbf{c}^*,\mathbf{s}^*)$ such that for every k, $\mathcal{D}_k^* = \tau_k(\mathbf{p}^*,\mathbf{c}^*)$, $p_k^* = \phi_k(\mathcal{D}_k^*, p_k^*, c_k^*, s_k^*)$, $c_k^* = \chi_k(\mathcal{D}_k^*, \mathbf{p}^*, s_k^*)$, and $s_k^* \in \omega_k(\mathcal{D}^*, \mathbf{p}^*, \mathbf{c}^*, \mathbf{s}_{-k}^*)$.

(ix) ***Equilibrium existence***: Finally, we will show that the fixed point satisfies conditions (I) – (III) in our equilibrium Definition 1. As (2) is derived from consumer utility maximization, condition (I) holds.

At the fixed point, (7) implies $\mathcal{Y}_k^* = \mathcal{D}_k^*$. If $c_k^* = \chi_k(\mathcal{D}_k^*, \mathbf{p}^*, s_k^*) < P_k$, we know $\mathcal{Y}_k^* = f_k^*$. So $\mathcal{D}_k^* = f_k^* \le \overline{f}_k$, and $\mathcal{D}_k^* = D_k^*$. If $c_k^* = P_k$, we have $p_k^* \ge P_k$, so $D_k^* = 0$. Then we have $\mathcal{D}_k^* = 0$, $\mathcal{Y}_k^* = 0$ and $f_k^* = 0$. Thus, every firm k's output, f_k^*, its expected demand, $\mathcal{Y}_k^*$, and actual demand, D_k^*, are all equal.

As consumers spend all income, the total income equals the total expenditure,

$$\sum_{i=1}^{m}(d_i^* + q_i^*) = \mathbf{p}^* \cdot \sum_{i=1}^{m} x_i^* \quad (10)$$

The total dividend equals the total profit, which is $\sum_{k=1}^{n} p_k^* D_k^*$ - $\mathbf{p}^* \sum_{k=1}^{n} z_k^*$ - $\sum_{k=1}^{n} h_k^*$. As $\sum_{k=1}^{n} p_k^* D_k^*$ - $\mathbf{p}^* \sum_{k=1}^{n} z_k^*$ = $\mathbf{p}^* \cdot \sum_{i=1}^{m} x_i^*$, (10) implies $\sum_{i=1}^{m} q_i^*$ =

$\sum_{k=1}^{n} h_k^*$, and the labor market is clear, as dictated by Walras law. Hence condition (III) holds.

To prove the fixed point satisfies condition (II), we need (see Appendix F):

Lemma 6: (i) If $f_k^* > 0$, p_k^* maximizes $(p_k - c_k^*)(D_k^* - s_k^* p_k^* + s_k^* p_k)$; (ii) If $f_k^* = 0$, $p_k^* f_k(\mathbf{z}_k, h_k) - \mathbf{p}_{-k}^* \cdot \mathbf{z}_k - h_k \le 0$ for any $(\mathbf{z}_k, h_k)$, and $D_k \le 0$ for any $p_k > p_k^*$.

Finally, we check the consistency between the production manager's decision and that of the marketing manager. Equations (3) and (4) imply that every firm k's inputs and labor demand maximize its profit given input prices and taking c_k^* as its output price. This implies $\Delta C_k/\Delta f_{k-} \le c_k^* \le \Delta C_k/\Delta f_{k+}$, where $\Delta C_k/\Delta f_{k-}$ and $\Delta C_k/\Delta f_{k+}$ are the left and right limits of the derivative of firm k's cost with respect to its output. On the other hand, Lemma 6 implies c_k^* is equal to the perceived marginal revenue, MR_k. So, we have $\Delta C_k/\Delta f_{k-} \le MR_k \le \Delta C_k/\Delta f_{k+}$. The first inequality implies a profit fall if firm k reduces its output, while the second inequality implies a profit fall if firm k increases its output. If the marginal cost exists, i.e., $\Delta C_k/\Delta f_{k+} = \Delta C_k/\Delta f_{k-}$, we get $MC_k = MR_k$. Overall the profit maximization condition (II) is achieved. Therefore, the fixed point is indeed generalized first-order equilibrium.

5. Equilibrium properties

Our model does not assume downward sloping demand curves. As shown in Appendix B, however, if $s_k^* > 0$, we must have $D_k^* = 0$. Thus, $f_k^* > 0$ implies $s_k^* < 0$. Upward sloping demand or perfectly inelastic demand never appears in equilibrium.

Property 1: Every active firm faces a downward sloping demand curve.

As we mentioned earlier, one of disadvantages of first-order equilibrium is that it may not contain Nash equilibrium or a local maximum equilibrium. One reason to introduce the generalized first-order equilibrium is to include any local maximum equilibrium. At a local maximum equilibrium, if a firm's demand is differentiable, obviously we can let the slope to be equal to the slope estimation and the equilibrium is also GFE. If the demand is not differentiable, we know that a price change in either direction will result in a decline in the profit. Then we can always find a slope estimation between the two limits of the demand derivatives such that a marginal price change will not increase the profit. Furthermore, if the cost is differentiable, we must have the equality between the marginal cost and perceived marginal revenue. The specific value of the marginal revenue then dictates a unique value of the slope estimation suitable for GFE. For instance, in both early examples of Bonnano and ours, there is a unique slope estimation under which GFE exists.

Property 2: A local maximum equilibrium is always a GFE. Moreover, if a firm's cost is differentiable, its slope estimation in the GFE is uniquely determined.

On the other hand, any GFE must be a Negishi equilibrium (1961), which does not impose any restriction on slope estimations. Hence our existence result implies the existence of Negishi equilibrium. An unsolved issue, however, is whether Negishi equilibrium exists with arbitrary slope estimations. More precisely, if we assign a negative and arbitrary number for each firm's slope estimation, will the previous equilibrium conditions (I) – (III) still hold, with "given $s_k^* < 0$" replacing "$s_k^* \in [\partial D_k^*/\partial p_k|_{+0}, \partial D_k^*/\partial p_k|_{-0}]$"? The answer is affirmative.

Property 3: Given any $s_k < 0$ for every firm k, there always exists a corresponding Negishi equilibrium (see Appendix G).

Since Negishi equilibrium exists not only for some slope estimations, but also for any negative estimations, the set of such equilibrium is obviously too large to bear any prediction regarding the sate of the economy.

Our model allows monopoly, oligopoly and monopolistically competitive markets the economy. We rule out perfect competition because homogeneous goods would violate the assumption of strictly quasi-concave utility and concave production functions. However, even with product differentiation, firms may have little market power when differentiated products are close substitutes. A robust test of our model would be to see if the equilibrium converges to that of perfect competition when product differentiation approaches zero.

In fact, if the products of two firm k and j are very similar, the derivatives of utility and production functions with respect to these products will almost be the same. The corresponding Hessian and boarded Hessian matrices will have two almost identical rows/columns. Their determinants will be close to zero. This implies a very large absolute value of the slope, s^*_k. In equilibrium, if $f^*_k > 0$, we have $p^*_k - c^*_k = -D^*_k/s^*_k$. As s^*_k is close to infinity, the price margin has to diminish.

Property 4: If firm k's product has at least one close substitute goods, its equilibrium output will approach the efficient level where price equals marginal cost.

If every firm has at least one rival producing close substitute goods, no firm would enjoy significant market power, and marginal cost pricing will prevail.

Furthermore, one can assume that every firm's slope estimation to be extremely large regardless of the true slope. Obviously, this will lead to marginal cost pricing and the economy approximates Arrow-Debreu's general equilibrium model with perfect competition, and almost reaches a Pareto efficient outcome.

5. Concluding remarks

Based on standard assumptions on consumer preference and technology, no equilibrium existence can be established in a general equilibrium model with imperfect competition and intermediate goods, except for possibly Negishi equilibrium. We introduce a generalized first-order equilibrium concept and have shown its existence based on assumptions of utility and production functions.

Needless to say, one short coming of this equilibrium is that some firms may not maximize their profits, locally or globally, despite their first-order condition for profit maximization hold. Unfortunately this embarrassment cannot be avoided when imperfect competition is involved. Our result shows how much one can obtain from the standard assumptions comparable to the general equilibrium model with perfect competition. Since Bonanno's example shows the equilibrium existence with higher level rationality is impossible, we have to accept the bounded rationality necessary for obtaining equilibrium in imperfect competition.

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Appendix A: (i) By Assumption 1, $\partial f_j/\partial z_{jk}$, is bounded and $\partial f_j/\partial h_j$ is bounded above zero. If $z_{jk} > 0$, we have $c_j\partial f_j/\partial z_{jk} = p_k$. By Assumption 3(i), we also have $h_j > 0$, so $c_j\partial f_j/\partial h_j = 1$. If $p_k > \sup.[(\partial f_j/\partial z_{jk})/(\partial f_j/\partial h_j)]$, we get $z_{jk} = 0$.

Similarly, by Assumption 2, $\partial u_i/\partial x_{ik}$ is bounded and $\partial u_i/\partial \ell_i$ is bounded above zero. If $x_{ik} > 0$, we have $\partial u_i/\partial x_{ik} = \lambda_i p_k$. The optimal q_i requires $\partial u_i/\partial \ell_i = \lambda_i$. If $p_k > \sup.[(\partial u_i/\partial x_{ik})/(\partial u_i/\partial \ell_i)]$, we get $x_{ik} = 0$.

So, if $p_k > P_k = \sup.[(\partial f_j/\partial z_{jk})/(\partial f_j/\partial h_j),(\partial u_i/\partial x_{ik})/(\partial u_i/\partial \ell_i)]$ for all j and i, $D_k = 0$.

(ii) For any $f_k(\mathbf{z}_k,h_k) > 0$, we have $c_k(\partial f_k/\partial h_k) = 1$. As $\partial f_k/\partial h_k$ is bounded, we let $C_k = \inf.(1/\partial f_k/\partial h_k)$. For $c_k \le C_k$, $c_k(\partial f_k/\partial h_k) \le 1$ for any h_k. As $\partial^2 f_k/\partial h_k^2 < 0$, we must have $h_k = 0$, and hence $f_k(\mathbf{z}_k,h_k) = 0$.

Appendix B: We consider all possible cases. (i) $s_k > 0$, we solve $\mathscr{p}_k = P_k$; (ii) $s_k = 0$. (ii.a) $\mathcal{D}_k > 0$, again we have $\mathscr{p}_k = P_k$; (ii.b) $\mathcal{D}_k = 0$, we get an interval $[c_k,P_k]$.

(iii) Suppose $s_k < 0$. (iii.a) $\mathcal{D}_k + s_k(c_k - p_k) < 0$, we obtain $\mathscr{p}_k = c_k$; (iii.b) $\mathcal{D}_k + s_k(c_k - p_k) \ge 0$ and $\mathcal{D}_k + s_k(2P_k - p_k - c_k) < 0$, we solve $\mathscr{p}_k = 0.5(p_k + c_k - \mathcal{D}_k/s_k)$; (iii.c) $\mathcal{D}_k + s_k(2P_k - p_k - c_k) \ge 0$, again we get $\mathscr{p}_k = P_k$.

In all of these cases, mapping (7) is either uniquely determined and continuous in $(\mathcal{D}_k,p_k,c_k,s_k)$, or becomes an interval $[c_k,P_k]$ in case (ii.b), where $\mathcal{D}_k = s_k = 0$. Apparently correspondence (7) is upper hemi-continuous.

Appendix C: In Appendix B we see $\mathcal{Y}_k$ is only affected by c_k in (iii.b), where it decreases in c_k. So $\mathcal{Y}_k$ is non-increasing in c_k. We now show $\mathscr{y}_k$ is strictly increasing in c_k for $\mathscr{y}_k > 0$. Let $\mathscr{y}_k$ be the optimal output associated with c_k, and suppose for $\Delta c_k > 0$, the optimal $\Delta\mathscr{y}_k < 0$. Then we must have $(c_k+\Delta c_k)(\mathscr{y}_k+\Delta\mathscr{y}_k) - c_k(\mathscr{y}_k+\Delta\mathscr{y}_k) \ge (c_k+\Delta c_k)\mathscr{y}_k - c_k(\mathscr{y}_k)$, where $c_k(\mathscr{y}_k)$ is the cost associated with $\mathscr{y}_k$. As $\Delta c_k\Delta\mathscr{y}_k < 0$, we get $c_k(\mathscr{y}_k+\Delta\mathscr{y}_k) - c_k(\mathscr{y}_k+\Delta\mathscr{y}_k) > c_k\mathscr{y}_k - c_k(\mathscr{y}_k)$, so $\mathscr{y}_k$ could not be the optimal output given c_k, a contradiction. Now suppose the optimal $\Delta\mathscr{y}_k = 0$. The strict concavity of $f(\mathbf{z}_k,h_k)$ implies there is no change in ($\mathbf{z}_k$,h_k). Then $(c_k+\Delta c_k)\partial f_k/\partial h_k = 1$ could not hold if $c_k\partial f_k/\partial h_k = 1$ does. Hence we have $\Delta\mathscr{f}_k < 0$, if $\mathscr{f}_k > 0$ and $\Delta c_k > 0$. As $\mathscr{f}_k$ is strictly increasing in c_k, so is $\mathcal{Y}_k - \mathscr{f}_k$.

From Lemma 1, we know $\mathscr{f}_k(\mathbf{p}_{-k},C_k) = 0$. If $\mathcal{Y}_k = 0$ at C_k, we have a unique solution for $\mathcal{Y}_k - \mathscr{f}_k = 0$ by Definition 5. If $\mathcal{Y}_k > 0$ at C_k, we start with $\mathcal{Y}_k - \mathscr{f}_k > 0$. As this function strictly increases in c_k, there is a unique solution of $\mathscr{c}_k$ either in (C_k,P_k) or equal to P_k. Function (8) must be continuous in $(\mathcal{D}_k,\mathbf{p},s_k)$ because $\mathcal{Y}_k$ is continuous in $(\mathcal{D}_k,p_k,s_k)$ and $\mathscr{f}_k$ is continuous in $\mathbf{p}_{-k}$.

Appendix D: (i) Prove $\partial z_{kj}/\partial p_j$ exists and is bounded: For convenience, we switch index k with j such that firm k is the input demander. Firm k's first-order condition for $(\mathbf{z}_k,h_k)$ is $c_k(\partial f_k/\partial \mathbf{z}_k) = \mathbf{p}_{-k}$, and $c_k(\partial f_k/\partial h_k) = 1$. Let $\mathbf{w}_k = (\mathbf{z}_k,h_k)$. We have $c_k(\partial f_k/\partial \mathbf{w}_k) = (\mathbf{p}_{-k},1)'$. Without loss of generality, we let p_j be the j-th element of $\mathbf{p}_{-k}$. Differentiate these equations with respect to p_j, we get:

$$c_k(\partial^2 f_k/\partial \mathbf{w}_k^2)(\partial \mathbf{w}_k/\partial p_j) + (\partial c_k/\partial p_j)(\partial f_k/\partial \mathbf{w}_k) = \tau \tag{D1}$$

where vector τ has the same dimension as $\mathbf{w}_k$, its j-th element is 1 and the rest zero.

Differentiate firm k's demand and supply balance condition $\mathcal{Y}_k(\mathcal{D}_k,p_k,c_k,s_k) - f_k[\mathbf{z}_k(\mathbf{p}_{-k},c_k),h_k(\mathbf{p}_{-k},c_k)] = 0$, with respect to p_j. We have:

$$(\partial\mathcal{Y}_k/\partial c_k)(\partial c_k/\partial p_j) - (f_k/\partial \mathbf{w})\cdot(\mathbf{w}_k/\partial p_j) = 0 \tag{D2}$$

We can write (D1) and (D2) together in a system of equations, whose total number is the dimension of τ plus 1.

$$\begin{bmatrix} c_k \dfrac{\partial^2 f_k}{\partial w_k^2} & \dfrac{\partial f_k}{\partial w_k} \\ \dfrac{\partial f_k}{\partial w_k}' & -\dfrac{\partial \Upsilon_k}{\partial c_k} \end{bmatrix} \begin{pmatrix} \dfrac{\partial w_k}{\partial p_j} \\ \dfrac{\partial c_k}{\partial p_j} \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix} \tag{D3}$$

As a partitioned matrix on the left-hand side of (D3), its determinant is equal to $c_k|\partial^2 f_k/\partial \mathbf{w}_k^2|\,|-\partial \mathcal{Y}_k/\partial c_k - (\partial f_k/\partial \mathbf{w}_k)'(c_k\partial^2 f_k/\partial \mathbf{w}_k^2)^{-1}(\partial f_k/\partial \mathbf{w}_k)|$. From Appendix B we know $\partial \mathcal{Y}_k/\partial c_k = 0$ in all cases except for (iii.b), where $\partial \mathcal{Y}_k/\partial c_k = 0.5s_k < 0$.

When $\partial \mathcal{Y}_k/\partial c_k = 0$, the determinant is equal to the determinant of the bordered Hessian matrix of $f_k(\mathbf{z}_k,h_k)$ multiplied by c_k. We know c_k is bounded away from zero. Then Assumption 1 ensures the determinant is bounded away from zero.

Now suppose $\partial \mathcal{Y}_k/\partial c_k = 0.5s_k < 0$. As matrix $(\partial^2 f_k/\partial \mathbf{w}_k^2)^{-1}$ is negative definite, and $\partial f_k/\partial \mathbf{w}_k$ has non-zero element, $\partial f_k/\partial h_k > 0$, so $(\partial f_k/\partial \mathbf{w}_k)'(\partial^2 f_k/\partial \mathbf{w}_k^2)^{-1}(\partial f_k/\partial \mathbf{w}_k) < 0$. Hence the absolute value of the determinant is larger than it is when $\partial \mathcal{Y}_k/\partial c_k = 0$, and must be bounded away from zero. The inverse of the matrix on the left hand side of (D3) exists. The solution of $\mathbf{w}_k/\partial p_j$, in particular, $\partial z_{kj}/\partial p_j$, exists, and is bounded.

(ii) Prove $\partial z_{kj}/\partial p_j < 0$ and is bounded away from zero: As f_k is concave, the leading principal minors of its Hessian matrix, $\partial^2 f_k/\mathbf{w}_k^2$, have alternating signs. We notice that the determinant of the matrix in (D3) has an opposite sign to $|\partial^2 f_k/\mathbf{w}_k^2|$. Thus this matrix must be negative semi-definite. We know $\partial z_{kj}/\partial p_j$ is equal to the j-th diagonal element of its inverse matrix.

For a non-singular negative semi-definite matrix, the product of any of its non-zero diagonal element and its counterpart in the inverse matrix is not less than 1, i.e., $c_k(\partial^2 f_k/z_{kj}^2)(\partial z_{kj}/\partial p_j) \geq 1$. As $c_k(\partial^2 f_k/z_{kj}^2) < 0$ and is bounded, $\partial z_{kj}/\partial p_j$ must be negative and bounded away from zero.

(iii) If no switch occurs for any input, and also no switch for $\mathcal{Y}_k/\partial c_k$ between zero and $0.5s_k$, $\partial z_{kj}/\partial p_j$ is continuous in arguments of $c_k\partial^2 f_k/\partial \mathbf{w}_k^2$, $\partial f_k/\partial \mathbf{w}_k$ and $\mathcal{Y}_k/\partial c_k$.

Appendix E: (i) We first show that (9) gives a negative value bounded away from zero in case (ii.a). Since we showed in Appendix D that every $\partial z_{jk}/\partial p_k$ is negative and bounded away from zero, it suffices to consider any consumer's demand, $\partial x_{ik}/\partial p_k$. As

$\mathcal{D}_k = 0$, the income effect is nil, $\partial x_{ik}/\partial p_k$ is equal to the slope of the Hicksian demand curve, $\partial x_{ik}^h/\partial p_k$, which is negative (see Barten and Boehm 1982).

To show $\partial x_{ik}^h/\partial p_k$ is bounded away from zero, we notice it is the k-th diagonal element of the inverse matrix of the bordered Hessian of $u_i(\mathbf{x}_i,\ell_i)$ (Barten and Boehm). Since $u_i(\mathbf{x}_i,\ell_i)$ is quasi-concave, its leading principle minors must have alternative signs (with weak inequalities), including its determinant. As the product of the determinant of a matrix and that of its inverse is always equal to 1, they must have the same sign. Since the determinant of the bordered Hessian is bounded away from zero, the sign of the determinant of the inverse of the bordered Hessian of $u_i(\mathbf{x}_i,\ell_i)$ satisfies the requirement of a negative semi-definite matrix (with a strict inequality).

On the other hand, the remaining leading principle minors of this inverse matrix are exactly those from the Slutsky matrix $\partial \mathbf{x}^h/\partial \mathbf{p}$, which is negative semi-definite. Therefore, the inverse of the bordered Hessian of $u_i(\mathbf{x}_i,\ell_i)$ is negative semi-definite. Finally, the product of its k's diagonal element and its counterpart in the bordered Hessian is no less than 1, i.e., $(\partial^2 u_i/x_{ik}^2)(\partial x_{ik}^h/\partial p_k) \geq 1$. As $\partial^2 u_i/x_{ik}^2 < 0$ and is bounded, $\partial x_{ik}^h/\partial p_k$ must be bounded away from zero.

(ii) Now we prove that in case of (ii.a), the switching point is unique. Suppose the opposite is true, i.e., there are two prices, $p_{k1} < p_{k2}$, such that $\mathcal{D}_k = 0$ at both prices, and $\mathcal{D}_k > 0$ for any $p_k \in (p_{k1},p_{k2})$. At p_{k1}, we must have $\partial\mathcal{D}_k/\partial p_k|_{-0} < 0$ as just shown above. By continuity, for the demand curve to reach another switching point p_{k2}, we must have some $\mathcal{D}_k > 0$ at the same price p_{k1}. Then, the demand is not unique at p_{k1}, violating the assumption of the strictly quasi-concave utility function and strictly concave production function.

(iii) Then, we show (9) is upper-hemi-continuous. In case (i) of Definition 6, (9) gives a single value if no switch occurs, and an interval otherwise; in (ii.a) we have a single value; in (ii.b) an interval; and in (ii.c) a single value. Since $\partial\mathcal{D}_k/\partial p_k$ is continuous except for at switching points, (9) must be a hemi-continuous correspondence throughout all cases.

Appendix F: (i) As shown in Appendix B, p^*_k may not maximize the profit function $(p_k–c^*_k)(D^*_k - s^*_k p^*_k + s^*_k p_k)$ only when p^*_k equals P_k, as in case (iii.c). But if $p^*_k = P_k$, we must have $D^*_k = 0$, a contradiction for $f^*_k > 0$. When $f^*_k > 0$, c^*_k equals the marginal cost, which is higher than the average cost given a strictly concave production function. On the other hand, c^*_k equals the marginal revenue, which is less than p^*_k as $s^*_k < 0$. So p^*_k is higher than the average cost, firm k's profit is positive.

(ii) In case of (ii.a) of Definition 6, we have $s^*_k < 0$. Correspondence (7) should yield a price lower than p^*_k. So p^*_k can be in the fixed point only in cases of (ii.b) and (ii.c), which means $D_k = 0$ if $p_k \geq c^*_k$. As the switching point is unique, we must have $D_k = 0$ for any $p_k > p^*_k$.

Appendix G: Our argument up to equation (6) in the text remains valid. Since $s_k < 0$, we need not consider cases (i) and (ii) in Appendix B. In case (iii), it has been shown that $\mathcal{p}_k$ always has a single value. In this case mapping (7) is a continuous function. Also, Lemma 3 applies to (8).

Given vector **s**, we have continuous functions (5'), (7) and (8): $\mathcal{D}_k = \tau_k(\mathbf{p},\mathbf{c})$, $\mathcal{p}_k = \phi_k(\mathcal{D}_k,p_k,c_k)$, $\mathcal{c}_k = \chi_k(\mathcal{D}_k,\mathbf{p})$. They together form a continuous function from a compact and convex set $[(\boldsymbol{\mathcal{D}},\mathbf{p},\mathbf{c})\in R^{3n}_{+} | 0 \leq \mathcal{D}_k \leq \overline{f}_k, C_k \leq p_k \leq P_k, C_k \leq c_k \leq P_k\}$ into itself. By Brouwer's fixed-point theorem, a fixed-point $(\boldsymbol{\mathcal{D}}^*,\mathbf{p}^*,\mathbf{c}^*)$ exists. Both Lemma 6 and equation (10) are valid, hence Negishi equilibrium exists for any negative vector **s**.