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On Input-Output Linkage Measures

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On input-output linkage measures*

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Abstract

We consider six widely used input-output (IO) linkage measures in both traditional and (less used) generalized IO settings. We derive closed-form expressions for the so-called hypothetical extraction linkages, which identify the exact driving forces behind these measures. Next, we propose a net forward linkage indicator as a "counterpart" of the Oosterhaven and Stelder's (2002) net backward linkage. It is shown that all the linkage measures are closely interrelated. Finally, we make an empirical assessment of similarity of all seven IO linkage measures.

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1 Introduction

In the fields of regional economics and development economics different measures have been proposed and extensively used to identify sectors with the highest potential of spreading growth impulses throughout the economy (see, e.g., Miller and Blair 2009, Chapter 12, and extensive references thereof). In this note we focus on these the so-called key sector measures, namely, backward and forward linkages, net backward linkages, and three linkage indicators that stem from the idea of a hypothetical extraction of sectors from an economy's production system. In particular, we provide closed-form analytical formulae for the last linkage measures, which identify the exact driving forces of this class of input-output (IO) linkages.[1]

However, in contrast to Miller and Blair (2009) and most IO linkage studies we do not only consider the setting with an exclusive focus on sectoral gross outputs, but also discuss the measures in a generalized IO setting. In the latter framework other economic, social, environmental or resource factors, such as income generation, emission of greenhouse gases, creating jobs, or water consumption, take the central stage in determining which industries are important. Further, we propose a new linkage indicator that is a "forward counterpart" of Oosterhaven and Stelder's (2002) net backward linkage measure. Finally, we show analytically and empirically how all the linkage indicators are interrelated. In order to make the empirical test meaningful, we work only with *normalized* linkage measures which are dimensionless and are given per unit of output or resource use.

2 Input-output linkage measures

Consider the open static Leontief model (see e.g., Miller and Blair 2009), given by $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{y}$, where $\mathbf{x}$ is the $n \times 1$ endogenous vector of gross outputs of n sectors, $\mathbf{A}$ is the $n \times n$ direct input requirements matrix, and $\mathbf{y}$ is the $n \times 1$ exogenous vector of final demands (including consumption, investments, exports, and government expenditures).[2] Denote the matrix of intermediate flows by $\mathbf{Z}$ with its generic entry

[1]These linkages are appropriate, for example, in examining the effect of complex production process on output decline (change) in transition economies. See e.g., Blanchard and Kremer (1997) who argue that a breakdown of complex chains of production caused by transition in former Soviet Union and Central European countries is one of the main factors explaining the rapid decline in their GDP from 1989 to 1994. Their empirical evidence suggests that output has fallen farthest for goods with the most complex production process.

[2]Adopting usual convention, matrices are given in bold uppercase letters; vectors in bold lowercase letters; and scalars in italic lowercase letters. Vectors are columns by definition, and transposition is indicated by a prime. $\hat{\mathbf{x}}$ denotes the diagonal matrix with the elements of the vector $\mathbf{x}$

z_{ij} being the value of deliveries from sector i to sector j. Then the input coefficients matrix is derived as $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ with its typical element a_{ij} denoting the output of industry i directly required as input for one unit of output in industry j. The reduced form of the Leontief model is

$$\mathbf{x} = \mathbf{L}\mathbf{y}, \tag{1}$$

where $\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1}$ is the Leontief inverse with $\mathbf{I}$ being the identity matrix (Leontief 1936, 1941). The typical element of the Leontief inverse, l_{ij}, denotes the output in industry i directly and indirectly required to satisfy one unit of final demand in industry j. The row vector of *output multipliers* is defined as $\mathbf{m}'_o = \boldsymbol{\imath}'\mathbf{L}$, where $\boldsymbol{\imath}$ is the summation vector of ones. Its i-th element $m^o_i = \sum_{k=1}^{n} l_{ki}$ indicates the increase of total output in all sectors per unit increase of final demand in sector i. As such it is also called the *total backward linkage* of sector i, which we correspondingly denote by

$$b_i = m^o_i = \sum_{k=1}^{n} l_{ki}. \tag{2}$$

From the input side, the accounting identity that holds each period is $\boldsymbol{\imath}'\mathbf{Z} + \mathbf{v}' = \mathbf{x}'$, where $\mathbf{v}$ is the total primary input vector (i.e., payments to labor, capital and imports). If one defines the matrix of output coefficients by $\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z}$, the mentioned identity can be written as $\mathbf{x}'\mathbf{B} + \mathbf{v}' = \mathbf{x}'$. Its reduced form is

$$\mathbf{x}' = \mathbf{v}'\mathbf{G}, \tag{3}$$

where $\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1}$ is the Ghosh inverse (Ghosh 1958). The typical element g_{ij} of $\mathbf{G}$ is interpreted as measuring the direct and indirect *value* increase of output in sector j due to a unit increase in price of the primary inputs in sector i. The i-th row sum of $\mathbf{G}$ is accordingly interpreted as the increase of the value of total output in all sectors per unit price increase of primary inputs in sector i.[3] As such the *total forward linkage* of sector i is defined as

on its main diagonal and zero elsewhere.

[3]We interpret the Ghosh model (3) as a *price* model according to Dietzenbacher (1997). De Mesnard 2009 further clarifies this interpretation by pointing out that the price changes can only be interpreted as a change in *relative* prices (or price index) for fixed outputs in current prices. Oosterhaven (1988, 1989, 1996) showed convincingly that the quantity interpretation of Ghosh model is entirely implausible. The older definition of forward linkages as row sums of the Leontief inverse (Rasmussen 1956) is considered inferior, as it actually measures the output effect of an economically senseless unit vector of final demand.

$$f_i = \sum_{k=1}^{n} g_{ik}. \quad (4)$$

There are some other linkage measures in the class of so-called *hypothetical extraction methods* (HEMs), which take their origin from the work of Paelinck et al. (1965), Strassert (1968) (as cited in Miller and Lahr 2001) and Schultz (1977). The central idea of the classical HEM in quantifying the importance of sectors to an economy is as follows. In order to estimate the importance of sector i, this sector is hypothetically eliminated from the production system, which allows to find its total contribution to the economy-wide output. Mathematically, one nullifies the i-th row and column of the input matrix $\mathbf{A}$ and the i-th element of the final demand vector $\mathbf{y}$, and then using (1) computes the reduced outputs in this hypothetical case. Denote the adjusted input matrix and final demand vector by $\mathbf{A}^{-i}$ and $\mathbf{y}^{-i}$, respectively. Then, the reduced outputs after extracting sector i are given by $\mathbf{x}^{-i} = \mathbf{L}^{-i}\mathbf{y}^{-i}$, where $\mathbf{L}^{-i} = (\mathbf{I} - \mathbf{A}^{-i})^{-1}$. The difference between total outputs of the economy before and after the extraction, $\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}^{-i}$, is called the *total linkage* of sector i and measures its importance to the economy. An alternative measure would be not include the original output of the sector in question, x_i. Thus, the alternative definition of the total linkage is $(\boldsymbol{\imath}'\mathbf{x} - x_i) - \boldsymbol{\imath}'\mathbf{x}^{-i}$. In both cases, division by total output $\boldsymbol{\imath}'\mathbf{x}$ gives the normalized total linkage measures that indicate the proportion of aggregate output reduction due to extraction of sector i.

Dietzenbacher and van der Linden (1997) further used a non-complete HEM to obtain backward and forward linkages. The idea is similar to the total linkage derivation, but now only a column of the input matrix $\mathbf{A}$ (resp. a row of the output matrix $\mathbf{B}$) is nullified in order to quantify the backward (resp. forward) linkage of the extracted sectors. If we assume that sector i buys no intermediate inputs from any production sectors, the input matrix becomes $\mathbf{A}_c^{-i}$ with zeros in the column corresponding to sector i. On the other hand, if all the intermediate sales of sector i are hypothetically eliminated, the corresponding matrix of output coefficients becomes $\mathbf{B}_r^{-i}$.[4] Then using the Leontief model (1) for backward linkage estimation and the Ghosh model (3) for forward linkage derivation, the corresponding linkages of sector i are obtained, respectively, from $\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}_c^{-i}$ and $\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}_r^{-i}$, where $\mathbf{x}_c^{-i} = (\mathbf{I} - \mathbf{A}_c^{-i})^{-1}\mathbf{y}$ and $(\mathbf{x}_r^{-i})' = \mathbf{v}'(\mathbf{I} - \mathbf{B}_r^{-i})^{-1}$.

[4]Note that while $\mathbf{A}^{-i}$ means that both the i-th row and column of the original input matrix are nullified, $\mathbf{A}_c^{-i}$ indicates that only *column* i is set to zero. For the forward linkage, similarly only the i-th *row* is nullified, hence the new output matrix is denoted by $\mathbf{B}_r^{-i}$.

However, comparing absolute HEM linkages is not very useful. They simply tell us that extracting big/small sectors tends to have big/small impacts on the economy. Moreover, since we want to do an empirical test of the different linkage measures, comparing the absolute HEM linkages with, for example, (2) and (4) is senseless, as the HEM linkages have a unit of measurement (say, thousands of euros) whereas the other linkages are dimensionless indicators (see Oosterhaven et al. 1986, p. 69). Hence, we will consider only *normalized* HEM linkages (multipliers) that are expressed per unit of output. The normalized hypothetical extraction backward and forward linkages are defined, respectively, as[5]

$$b_i^h = \frac{\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}_c^{-i}}{x_i} \quad \text{and} \quad f_i^h = \frac{\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}_r^{-i}}{x_i}. \tag{5}$$

The superscript h is used to distinguish these hypothetical extraction linkages from the corresponding linkages in (2) and (4), respectively.

A more recent measure for key sectors was proposed by Oosterhaven and Stelder (2002), which raised a hot debate (for referecences see Miller and Blair 2009, p. 282) and in the end it was agreed to be labeled as a *net backward linkage*. Sector i's net backward linkage is defined as

$$b_i^n = \frac{b_i y_i}{x_i}, \tag{6}$$

i.e., it is the ratio of the output generated in all sectors due to the final demand of sector i and the output generated in sector i for the satisfaction of the final demands of all industries. Thus, a key sector i with $b_i^n > 1$ implies that the rest of the economy is more dependent on sector i than that sector i is dependent on the rest of the economy. This emphasizes that the net linkage (6), as opposed to the other linkages, takes the two-sided nature of sectoral dependency into (full) account.

3 Our contribution

In what follows we focus on the interrelationships of the above mentioned linkage measures. In particular, we want to derive the closed form expressions for the hypothetical extraction linkage indicators, as this makes the convoluted extraction procedure redundant and allows for an analytical comparison with the other measures.

[5]Other different normalizations are possible, such as with regard to overall gross output. See Table 12.5 in Miller and Blair (2009, p. 565) that summarizes different 'normalized' HEM linkages. However, for our empirical test purposes normalization with respect to sectoral outputs, x_i, is most appropriate, since it avoids artificial high correlation between different IO linkages.

Such an investigation has been done with respect to the total linkage in a more general setting recently by Temurshoev (2010), who considers the effect of the extraction of one sector and a group of sectors on any economic, social and/or environmental factor(s). He terms the total linkage of sector i (i.e., $\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}^{-i}$) as the *gross output worth* of i and proves that its closed form formula equals $\tilde{\omega}_i^o = m_i^o x_i / l_{ii} = b_i x_i / l_{ii}$ (Temurshoev 2010, Theorem 1, footnote 6). The closed form expression of $\tilde{\omega}_i^o$ shows that the output worth of sector i according to the HEM approach is not only dependent on its output multiplier or backward linkage, b_i, but also on the size of the sector's output x_i and its total self-dependency on inputs as indicated by l_{ii}. This makes sense since a sector with the large output multiplier and large total output is contributing much to the aggregate output, but a sector that is largely dependent on itself gains lower economy-wide importance as it will have less potential of spreading exogenous stimulus throughout the economy. Thus, a sector with the maximum gross output worth is a key sector from the HEM perspective.

Since we are primary interested in a dimensionless indicator, instead of $\tilde{\omega}_i^o$, the *normalized* gross output worth of sector i is used which simply equals $\omega_i^o = \tilde{\omega}_i^o / x_i$, i.e.,

$$\omega_i^o = \frac{m_i^o}{l_{ii}} = \frac{b_i}{l_{ii}}. \tag{7}$$

Equation (7) shows that performing the traditional three-step procedure in order to derive sectors' (normalized) output worths is redundant, which may become a formidable task when the number of sectors is large.

The interesting next question is whether the closed-form expressions for the noncomplete hypothetical extraction linkages given in (5) are comparably simple, and the answer is yes. In the appendix we prove the following result.[6]

Result 1. *The closed-form expressions of the backward and forward linkages resulting from the hypothetical extraction of sector i are given, respectively, by*

$$b_i^h = \frac{b_i - 1}{l_{ii}} \quad \textit{and} \quad f_i^h = \frac{f_i - 1}{l_{ii}}. \tag{8}$$

There are two implications of expressions (8). First, an analyst in deriving the backward and forward linkages according to the noncomplete HEM method does *not* need to extract any sector, and can simply use the formulas given in Result 1.

[6]The absolute noncomplete HEM linkages can be easily derived from Result 1. For example, $\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}_r^{-i} = b_i^h \times x_i = (b_i - 1)x_i / l_{ii}$ which is expressed in some measurement units. Note that in the second expression of (8) we have used the fact that $g_{ii} = l_{ii}$ (see the appendix).

Second, observe that

$$\omega_i^o = b_i^h + \frac{1}{l_{ii}},$$

which together with (7) implies that whenever the ratio $1/l_{ii}$ is identical for all i, then the result of three linkage measures, namely, total backward linkage (2), normalized gross output worth (7) and the HEM backward linkage (8) are exactly equivalent in terms of the rankings of sectors' importance in generating gross output.[7] This is, however, rarely observed in real life. Thus, in general, the outcomes of the three indicators are different.

The ratio $1/l_{ii}$, which is not taken into account by the total backward linkage b_i, is an important factor, because it corrects the linkage size of the extracted sector for its self-dependency. This correction is crucial for understanding the economy-wide impact of a sector. Also it is interesting to note that the factor $1/l_{ii}$ is added multiplicatively to the output multiplier of sector i, b_i, to arrive at the normalized gross output worth measure (7), whereas it is entered additively to b_i^h to give the normalized gross output worth of sector i. Thus, the degree of self-dependency of any sector is only partially (and differently) taken into account by b_i and b_i^h.

Incorporation of other factors (besides gross output) into the IO framework, such as economic, environmental and resource factors, leads to the generalization of the discussed linkages. Let the vector of *direct factor coefficients* $\boldsymbol{\pi}$ denote the sectoral factor usage/production per unit of total output. The row vector of *factor multipliers* is then given by $\mathbf{m}'_\pi = \boldsymbol{\pi}'\mathbf{L}$, where its i-th element $m_i^\pi = \sum_{k=1}^n \pi_k l_{ki}$ indicates the economy-wide increase of factor usage/production per unit increase of final demand in industry i. Hence, we denote the *total factor backward linkage* of sector i by

$$b_i^\pi = m_i^\pi = \sum_{k=1}^{n} \pi_k l_{ki}. \tag{9}$$

Alternatively, take the vector $\mathbf{G}\boldsymbol{\pi}$. Its i-th element $\sum_{k=1}^n g_{ik}\pi_k$ indicates an increase in the value of factor usage of all sectors per unit price increase of primary input of sector i. Thus, it is also the *total factor forward linkage* of sector i and is denoted as

$$f_i^\pi = \sum_{k=1}^{n} g_{ik}\pi_k. \tag{10}$$

Note that if $\pi_k = 1$ for all $k = 1, \ldots, n$, then the total factor backward and for-

[7] In such case, also the outcome of total forward linkage (4) and the HEM forward linkage in (8) match.

ward linkages in (9) and (10) boil down to their gross output counterparts given, respectively, in (2) and (4). That is why Lenzen (2003, p. 10) calls (9) and (10) generalized backward and forward linkages. Using the Leontief and Ghosh models in (1) and (3), respectively, note that

$$\mathbf{b}_\pi'\mathbf{y} = \boldsymbol{\pi}'\mathbf{L}\mathbf{y} = \boldsymbol{\pi}'\mathbf{x} = \mathbf{x}'\boldsymbol{\pi} = \mathbf{v}'\mathbf{G}\boldsymbol{\pi} = \mathbf{v}'\mathbf{f}_\pi,$$

that is, the total factor requirement needed to satisfy final demand $\mathbf{y}$ is equal to total factor usage accompanying sectoral primary inputs $\mathbf{v}$ (Oosterhaven 1996).

The HEM linkages can be also reformulated in this more general setting. Firstly, it has been shown by Temurshoev (2010) that the *factor worth* of sector i gives the analytical expression for the reduction in factor generation due to hypothetical elimination of sector i from the production system, $\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i}$, and equals $\tilde{\omega}_i^\pi = m_i^\pi x_i/l_{ii} = b_i^\pi x_i/l_{ii}$. Dividing $\tilde{\omega}_i^\pi$ by the total amount of factor produced/used by industry i, $\pi_i x_i$, gives the *normalized* factor worth of sector i as

$$\omega_i^\pi = \frac{m_i^\pi}{\pi_i l_{ii}} = \frac{b_i^\pi}{\pi_i l_{ii}}. \tag{11}$$

This is a generalization of the normalized output worth indicator (7) to a setting where one is mainly interested in some other factors (say, income or CO_2 emissions) rather than gross output.

If we replace the summation vector in (5) by the vector of direct factor coefficients and normalize with respect to $\pi_i x_i$, we obtain the hypothetical extraction *factor* backward and forward linkages as

$$b_i^{\pi,h} = \frac{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}_c^{-i}}{\pi_i x_i} \quad \text{and} \quad f_i^{\pi,h} = \frac{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}_r^{-i}}{\pi_i x_i}. \tag{12}$$

These measures again ask for a three-step calculation procedure: (i) nullifying a column of $\mathbf{A}$ or a row of $\mathbf{B}$, (ii) estimating the reduced outputs/inputs in the hypothetical case using Leontief or Ghosh models, and (iii) finding the corresponding differences given in (12). However, these steps are rather excessive, since similar to Result 1, the following closed-form expressions for $b_i^{\pi,h}$ and $f_i^{\pi,h}$ can be obtained.

Result 2. *The closed-form expressions of the hypothetical extraction factor backward and forward linkages of sector i are given, respectively, by*

$$b_i^{\pi,h} = \frac{b_i^\pi - \pi_i}{\pi_i l_{ii}} \quad \textit{and} \quad f_i^{\pi,h} = \frac{f_i^\pi - \pi_i}{\pi_i l_{ii}}. \tag{13}$$

Note that Result 1 is a special case of Result 2 when $\pi_i = 1$ for all i. Then, as before the relation between the normalized factor worth and HEM factor backward linkage for sector i is

$$\omega_i^{\pi} = b_i^{\pi,h} + \frac{1}{l_{ii}}.$$

Thus, input dependency of sector i on itself is partially considered in $b_i^{\pi,h}$, while adding to it the term $1/l_{ii}$ fully considers i's self-dependency.

It needs to be mentioned that the HEM can be applied to more than one sector, which for the case of complete elimination of a group of industries from the production system is thoroughly explored in Temurshoev (2010). This can be also done with respect to non-complete hypothetical extraction of only purchases or only sales of group of sectors that would result in the hypothetical extraction *group* backward and forward linkage measures. We, however, do not pursue such aim in this note, because we think that for measuring *economy-wide* impact of a sector or group of sectors the complete extraction is the most adequate HEM approach. But there might exist other settings where the non-complete HEM approach is more appropriate (see e.g., footnote 1), hence we derived their corresponding closed-form formulations.

Oosterhaven and Stelder's (2002) net backward linkage of sector i in the generalized IO framework, as originally proposed by the authors is defined as[8]

$$b_i^{\pi,n} = \frac{b_i^{\pi} y_i}{\pi_i x_i}. \tag{14}$$

In the present context we refer to (14) as the *factor net backward linkage*. It equals the ratio of the amount of factor generated in all sectors due to the final demand of sector i and the factor generated in sector i due to the final demands of all industries. Thus, a key sector i with $b_i^{\pi,n} > 1$ implies that the economy-wide factor generated by the final demand of sector i is larger than the amount of sector i's factor that is generated by all industries' (including i's) final demands. This emphasizes the two-sided nature of sectoral dependency.

Finally, it is interesting to find the "forward counterpart" of Oosterhaven and Stelder's net backward linkage measure. Given the interpretation of the last indicator, it is not difficult to derive the *factor net forward linkage*, which we define for sector i as[9]

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[8]In matrix form, the row vector of factor net backward linkages is given by $(\mathbf{b}_{\pi}^{n})' = (\boldsymbol{\pi}'\mathbf{L}\hat{\mathbf{y}})(\widehat{\hat{\boldsymbol{\pi}}\mathbf{L}\mathbf{y}})^{-1} = (\mathbf{b}_{\pi}'\hat{\mathbf{y}})(\widehat{\hat{\boldsymbol{\pi}}\mathbf{x}})^{-1}$.

[9]In matrix form, the vector of factor net forward linkages is equal to $\mathbf{f}_{\pi}^{n} = (\widehat{\mathbf{v}'\mathbf{G}\hat{\boldsymbol{\pi}}})^{-1}(\hat{\mathbf{v}}\mathbf{G}\boldsymbol{\pi}) = (\widehat{\mathbf{x}'\hat{\boldsymbol{\pi}}})^{-1}(\hat{\mathbf{v}}\mathbf{f}_{\pi})$.

$$f_i^{\pi,n} = \frac{v_i f_i^{\pi}}{\pi_i x_i}, \tag{15}$$

i.e., it is the ratio of the amount of factor usage by all sectors associated with the value of primary inputs of sector i and the factor usage by sector i accompanying the value of primary inputs of all industries. Thus, a key 'forward' sector i with $f_i^{\pi,n} > 1$ implies that the economy-wide factor usage associated with the value-added of sector i is larger than the amount of sector i's factor that is accompanying the primary inputs of all industries. In this sense, sector i is more important for other sectors than other industries are for sector i.

Apparently, in the traditional gross output approach, $\pi_i = 1$ for i, thus (15) boils down to the *net forward linkage* measure

$$f_i^n = \frac{v_i f_i}{x_i}.$$

Since the Leontief and Ghosh systems are interrelated (see the appendix) and all the considered linkage indicators are based on these two models, the next issue we address is to search for (some) explicit mathematical relations that characterize such interdependencies among the linkage measures. Let the shares of sector i's final demand and primary inputs in its total output be $s_i = f_i/x_i$ and $\tilde{s}_i = v_i/x_i$, respectively. Then,

Result 3. *The following identities between the linkage measures always hold:*

$$b_i^{\pi,h} = \frac{\omega_i^{\pi}(b_i^{\pi} - \pi_i)}{b_i^{\pi}} = \frac{\omega_i^{\pi}(b_i^{\pi,n} - s_i)}{b_i^{\pi,n}} = \omega_i^{\pi}\left(1 - \frac{f_i^{\pi}\tilde{s}_i}{b_i^{\pi} f_i^{\pi,n}}\right), \quad \textit{and} \tag{16}$$

$$f_i^{\pi,h} = \frac{\omega_i^{\pi}(f_i^{\pi} - \pi_i)}{b_i^{\pi}} = \frac{\omega_i^{\pi}(f_i^{\pi} - \pi_i)}{\pi_i b_i^{\pi,n}} s_i = \frac{f_i^{\pi}}{f_i^{\pi,n}} \frac{\omega_i^{\pi}}{b_i^{\pi}} \frac{f_i^{\pi} - \pi_i}{\pi_i} \tilde{s}_i, \tag{17}$$

where $s_i = f_i/x_i$ and $\tilde{s}_i = v_i/x_i$ are, respectively, the shares of sector i's final demand and primary inputs in its gross output.

Result 3 clearly indicates that all the linkage indicators are related to one another. The equivalent relations among output linkages can be easily obtained by setting $\pi_i = 1$.[10] Identities (16) and (17) implies that knowledge about some of the linkages can be effectively used to derive other linkage indicators.

[10]One can also write the corresponding relations in terms of changes in linkage indicators. For example, taking total differential of the first expression in (16) in case of output yields $db_i^h = \frac{b_i - 1}{b_i} d\omega_i^o + \frac{\omega_i^o}{b_i^2} db_i$. Note that we cannot make the usual comparative static analysis (e.g., fix $db_i = 0$ and see how db_i^h and $d\omega_i^o$ are interrelated) as a change in one linkage measure implies simultaneous change in *all* other IO linkages.

4 Empirical similarities

In an attempt to find out how similar (or different) the outcomes of the linkage indicators are, we used the OECD IO database to compute all seven linkage measures for six countries (Austria, Greece, India, Indonesia, Japan, and the USA) and two time periods. We consider both the gross output approach and a generalized setting where the factor of interest is income. The matrices of Spearman's correlation coefficients are given in Table 1, where the lower diagonal elements correspond to the correlations between linkages for the first indicated year (say, 1995) and the upper diagonal entries are those for the later year (say, 2000). For the sake of readability, depending on the values of the correlation coefficients circles were added and should read as follows: circles with larger black (resp. white) area visualize stronger positive (resp. negative) correlations.

Table 1 is not very informative about the *overall* relationships between IO linkages across countries. For example, in case of gross output linkages, total and net backward linkages (i.e., b_i and b_i^n) are strongly positively correlated in India (i.e., the rank correlations are equal to 0.43 and 0.56 in the two considered periods), while there is no-any association between the two linkages in Greece (the corresponding rank correlations are -0.06 and -0.09). These differences more or less reflect the different structures of the two countries. We should also note that the correlation between two linkages can change (significantly) between the two periods for the same country. For example, in the US in 1995 the Spearman rank correlation between total income backward linkage, b_i^π, and hypothetical extraction income backward linkage, $b_i^{\pi,h}$, was equal to -0.39, while becomes 0.31 in 2000. Thus, we observe strong negative association between the mentioned linkages in 1995, but that significantly changes to strong positive relation in 2000.

The overall picture of the IO linkages' association is summarized in Table 2, which provides the averages and standard deviations of the tables of Spearman rank correlations from Table 1. The above-diagonal (resp. below-diagonal) elements correspond to output (resp. income) linkage associations. We observe that, for example, all three types of forward linkages are mutually highly positively correlated in case of both output and income linkages.[11] The same holds for output/income worths and hypothetical extraction backward linkages. Further, we find a strong

[11] Very high correlations between income forward f_i^π and income net forward linkages $f_i^{\pi,n}$ is mainly explained by the fact that income (value-added) composed the major part of the difference between gross and intermediate outputs. This can be seen from (15), which boils down to $f_i^{\pi,n} = f_i^\pi$ whenever $v_i = \pi_i x_i$.

Table 1: Spearman's rank correlation coefficients for seven IO linkages

	B	BH	BN	F	FH	FN	W	B	BH	BN	F	FH	FN	W
	Rank correlations of gross output IO linkages													
	Austria 1995 and 2000							Greece 1995 and2000						
B		0.94	0.23	0.53	0.50	-0.01	0.82		0.96	-0.09	0.64	0.64	0.23	0.86
BH	0.94		0.24	0.51	0.51	-0.02	0.95	0.97		-0.01	0.58	0.60	0.17	0.95
BN	0.31	0.29		-0.38	-0.39	-0.24	0.25	-0.06	0.04		-0.59	-0.59	-0.50	0.10
F	0.52	0.51	-0.38		0.99	0.72	0.44	0.62	0.55	-0.57		0.99	0.82	0.48
FH	0.45	0.48	-0.41	0.98		0.74	0.48	0.61	0.54	-0.56	0.99		0.82	0.52
FN	0.00	0.02	-0.36	0.77	0.81		0.02	0.13	0.05	-0.40	0.77	0.78		0.10
W	0.84	0.96	0.27	0.46	0.48	0.06		0.91	0.97	0.15	0.46	0.47	0.00	
	India 1993-94 and 1998-99							Indonesia 1995 and 2000						
B		0.96	0.56	0.30	0.28	-0.16	0.91		0.84	0.16	0.00	-0.05	-0.58	0.63
BH	0.94		0.65	0.21	0.21	-0.22	0.98	0.89		0.28	-0.17	-0.14	-0.72	0.93
BN	0.43	0.56		-0.35	-0.35	-0.29	0.67	-0.17	-0.07		-0.93	-0.93	-0.79	0.31
F	0.44	0.34	-0.32		0.99	0.73	0.19	0.32	0.20	-0.94		0.98	0.72	-0.23
FH	0.40	0.32	-0.32	1.00		0.74	0.19	0.25	0.18	-0.94	0.98		0.72	-0.15
FN	-0.14	-0.21	-0.21	0.63	0.65		-0.21	-0.49	-0.58	-0.60	0.52	0.55		-0.68
W	0.86	0.97	0.61	0.28	0.27	-0.20		0.64	0.90	0.00	0.11	0.14	-0.57	
	Japan 1995 and 2000							USA 1995 and 2000						
B		0.89	0.11	0.14	0.00	-0.63	0.68		0.96	0.38	0.58	0.53	0.24	0.84
BH	0.91		0.14	0.03	0.00	-0.67	0.93	0.93		0.48	0.49	0.46	0.18	0.95
BN	0.14	0.26		-0.90	-0.91	-0.69	0.14	-0.06	0.09		-0.16	-0.15	-0.04	0.56
F	0.07	-0.10	-0.89		0.96	0.58	-0.03	0.37	0.20	-0.90		0.99	0.84	0.40
FH	-0.04	-0.13	-0.89	0.97		0.65	0.02	0.28	0.17	-0.91	0.98		0.88	0.41
FN	-0.58	-0.66	-0.79	0.73	0.79		-0.58	-0.37	-0.46	-0.84	0.68	0.73		0.19
W	0.61	0.87	0.27	-0.20	-0.13	-0.55		0.69	0.90	0.25	-0.01	0.03	-0.49	
	Rank correlations of income IO linkages													
	Austria 1995 and 2000							Greece 1995 and2000						
B		0.23	0.34	0.35	0.26	0.32	0.24		-0.18	0.39	0.20	-0.07	0.07	-0.17
BH	0.40		0.30	0.20	0.68	0.20	0.99	-0.14		0.05	0.33	0.75	0.44	0.99
BN	0.23	0.39		-0.40	-0.28	-0.40	0.33	0.51	0.01		-0.58	-0.41	-0.57	0.12
F	0.34	0.14	-0.42		0.79	1.00	0.18	0.16	0.31	-0.56		0.79	0.98	0.29
FH	0.38	0.65	-0.26	0.77		0.80	0.66	-0.09	0.74	-0.46	0.81		0.87	0.71
FN	0.30	0.12	-0.42	1.00	0.76		0.18	0.10	0.37	-0.57	0.99	0.84		0.40
W	0.40	0.99	0.39	0.14	0.65	0.13		-0.12	0.99	0.06	0.28	0.71	0.34	
	India 1993-94 and 1998-99							Indonesia 1995 and 2000						
B		-0.14	0.06	0.65	-0.02	0.51	-0.13		-0.50	0.01	0.20	-0.35	0.09	-0.44
BH	-0.18		0.65	-0.17	0.60	-0.06	1.00	-0.65		0.36	-0.51	0.30	-0.45	0.98
BN	0.02	0.57		-0.38	-0.02	-0.38	0.67	0.12	0.04		-0.91	-0.66	-0.93	0.38
F	0.69	-0.12	-0.36		0.46	0.97	-0.16	0.11	-0.27	-0.87		0.60	0.98	-0.51
FH	-0.02	0.68	0.00	0.45		0.57	0.60	-0.50	0.55	-0.69	0.57		0.67	0.28
FN	0.50	0.07	-0.31	0.93	0.63		-0.06	0.03	-0.13	-0.87	0.97	0.67		-0.46
W	-0.17	0.99	0.60	-0.14	0.65	0.05		-0.57	0.98	0.06	-0.29	0.52	-0.16	
	Japan 1995 and 2000							USA 1995 and 2000						
B		-0.56	-0.03	0.27	-0.40	0.10	-0.56		0.31	0.34	0.54	0.37	0.48	0.33
BH	0.18		0.33	-0.38	0.40	-0.28	0.98	-0.39		0.50	0.33	0.68	0.34	0.99
BN	0.38	0.28		-0.82	-0.57	-0.77	0.35	0.05	0.20		-0.14	-0.02	-0.13	0.55
F	-0.39	-0.44	-0.89		0.57	0.94	-0.40	-0.02	-0.18	-0.95		0.84	0.99	0.29
FH	-0.35	0.23	-0.77	0.73		0.69	0.40	-0.34	0.52	-0.59	0.67		0.85	0.63
FN	-0.39	-0.44	-0.89	1.00	0.73		-0.27	-0.11	-0.16	-0.94	0.98	0.69		0.30
W	0.18	0.97	0.32	-0.48	0.17	-0.48		-0.37	0.97	0.26	-0.25	0.45	-0.23	

Note: Abbreviations are: B – backward, F – forward, H – hypothetical extraction, N – net, W – worth. Entries below (resp. above) the diagonal of the correlation matrix give the rank correlations for the first (resp. second) indicated year. *The OECD Input-Output Database* (2006 edition) was used to compute linkages for 48 industries (commodities in case of Indonesia). All correlations of at least 0.29 in absolute value are statistically significant at 5% and/or 1% level. Circles with larger black (white) area visualize stronger positive (negative) associations.

negative association (of an order of -0.61) between output/income net backward linkages and total output/income forward linkages.

To visualize the results, we use a *hierarchical agglomerative cluster analysis* (HCA) to identify groups of the IO linkages that are most similar in their outcomes (for HCA see e.g., Lattin et al. 2003, Chapter 8). As the basis similarity matrix we

Table 2: Summary of Spearman's rank correlations tables

	B	BH	BN	F	FH	FN	W
B		● 0.93 (0.04)	◑ 0.16 (0.23)	◕ 0.38 (0.22)	◕ 0.32 (0.25)	◑ -0.20 (0.32)	● 0.77 (0.11)
BH	◑ -0.13 (0.35)		◑ 0.24 (0.23)	◑ 0.28 (0.26)	◑ 0.27 (0.26)	◑ -0.26 (0.34)	● 0.94 (0.03)
BN	◑ 0.20 (0.19)	◕ 0.31 (0.21)		○ -0.61 (0.29)	○ -0.61 (0.29)	◔ -0.48 (0.26)	◕ 0.30 (0.21)
F	◑ 0.26 (0.30)	◑ -0.06 (0.31)	○ -0.61 (0.27)		● 0.98 (0.01)	● 0.71 (0.09)	◑ 0.20 (0.26)
FH	◑ -0.10 (0.30)	● 0.56 (0.17)	◔ -0.39 (0.28)	● 0.67 (0.14)		● 0.74 (0.09)	◑ 0.25 (0.25)
FN	◑ 0.17 (0.27)	◑ 0.00 (0.30)	○ -0.60 (0.28)	● 0.98 (0.02)	● 0.73 (0.09)		◑ -0.24 (0.31)
W	◑ -0.11 (0.34)	● 0.98 (0.01)	◕ 0.34 (0.20)	◑ -0.09 (0.31)	◕ 0.54 (0.18)	◑ -0.02 (0.30)	

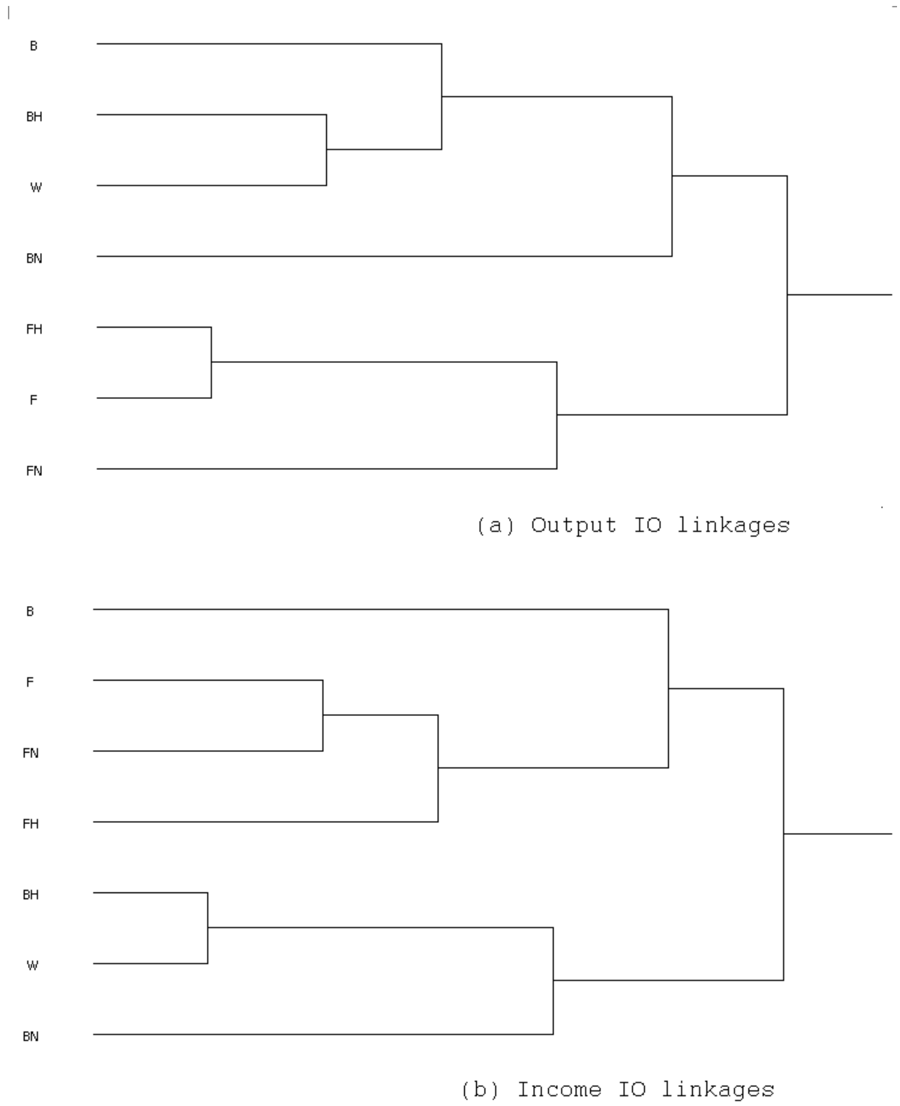
Note: Entries below (resp. above) the diagonal of the table give the *average* of $n = 12$ corresponding Spearman rank correlations from Table 1 for income (resp. output) linkages. Standard errors are given in parentheses. For other notations, see notes to Table 1.

use average correlations in Table 2. Next, the HCA starts from $n = 7$ clusters of size 1 and at each stage of the process finds the two "closest" (most homogeneous) clusters and joins them together. Then on the basis of new similarity matrices this process continues until only one cluster of size n remains. This hierarchical sequence of merging clusters is visually depicted by a tree diagram, also called a *dendrogram*. We have used the (weighted) *average link* criteria for forming clusters, which computes the similarity of the average scores in the newly formed cluster to all other partitions. The resulting tree diagrams for output and income linkages are given in Figure 1.[12]

From Figure 1 it is clear that higher values of agglomeration indicate less similarity or greater within-cluster "distance". If we start with a high level of agglomeration, we observe that there are two clusters of linkages. In case of output these are the set of all three types of forward linkages {F,FH,FN} and the rest comprising the second cluster, whereas in case of income linkages, total income backward linkage, B, joins the forward linkage group. A lower agglomeration level (or, higher similarity level) results in three clusters: {B,BH,W}, {BN} and {F,FH,FN} for output linkage measures, and {B}, {BH,BN,W}, and {F,FH,FN} for income linkages. That is, now compared to the previous case, net backward linkage constitutes an extra separate cluster for output linkages, while this role is delegated to the total backward linkage in case of income linkages. Still lower agglomeration level resulting in four clusters of "similar" measures, separates the net (resp. hypothetical extraction) forward linkage as one cluster in case of output (resp. income) linkages. Of course, choosing a level of agglomeration that provides the "best" representation of the number

[12]We used UCINET software for our cluster analysis (see Borgatti et al. 2002).

Figure 1: Hierarchical dendrograms for output and income IO linkages



of similar linkages is somewhat subjective. However, what we clearly see from our findings is the following:

1. Sectors' hypothetical backward linkages (BH) and worth measures (W) are the most similar linkages for both gross output and income.
2. Net backward linkage (BN) is quite different from all other measures. The same holds for the net output forward linkage.[13]

We note, however, that a high correlation does not necessarily mean totally equivalent outcomes of the linkages, which in fact are often different if one considers sectors

[13]The reason for closeness of the net income linkage (FN) and the total forward linkage (F) has been already pointed out in footnote 11. Thus, it might very well be the case that for other factors than income, FN shows totally different outcome too.

in the top positions of the derived rankings. Other than that our above-mentioned two observations are useful to practitioners in that they clearly give an indication of the most similar and most different linkages, and thus an analyst can choose the most appropriate measure(s) for his/her own research purposes.

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Appendix

Derivation of (8). Denote the i-th column of the identity matrix by $\mathbf{e}_i$. Then it is easy to confirm that $\mathbf{A}_c^{-i} = \mathbf{A}(\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')$. In order to derive the reduced outputs, we make use of the following identity for any nonsingular matrix $\mathbf{X}$ and any vectors $\mathbf{u}$ and $\mathbf{v}$ (see Henderson and Searle 1981, p. 53):

$$(\mathbf{X} + \mathbf{u}\mathbf{z}')^{-1} = \mathbf{X}^{-1} - \frac{1}{1 + \mathbf{z}'\mathbf{X}^{-1}\mathbf{u}}\mathbf{X}^{-1}\mathbf{u}\mathbf{z}'\mathbf{X}^{-1}. \tag{A1}$$

That is, for $\mathbf{X} = \mathbf{I} - \mathbf{A}$, $\mathbf{u} = \mathbf{A}\mathbf{e}_i$ and $\mathbf{z} = \mathbf{e}_i$, we have

$$\begin{aligned}\mathbf{x}_c^{-i} &= (\mathbf{I} - \mathbf{A}_c^{-i})^{-1}\mathbf{y} = (\mathbf{I} - \mathbf{A} + \mathbf{A}\mathbf{e}_i\mathbf{e}_i')^{-1}\mathbf{y} = \left(\mathbf{L} - \frac{1}{1 + \mathbf{e}_i'\mathbf{L}\mathbf{A}\mathbf{e}_i}\mathbf{L}\mathbf{A}\mathbf{e}_i\mathbf{e}_i'\mathbf{L}\right)\mathbf{y} \\ &= \mathbf{x} - \frac{1}{1 + \mathbf{e}_i'(\mathbf{L} - \mathbf{I})\mathbf{e}_i}(\mathbf{L} - \mathbf{I})\mathbf{e}_i\mathbf{e}_i'\mathbf{x} = \mathbf{x} - \frac{x_i}{l_{ii}}(\mathbf{L} - \mathbf{I})\mathbf{e}_i,\end{aligned}$$

where we have used the fact that $\mathbf{L}\mathbf{A} = \mathbf{A} + \mathbf{A}^2 + \cdots = \mathbf{L} - \mathbf{I}$. Hence, the hypothetical extraction backward linkage of sector i is equal to

$$b_i^h = \frac{\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}_c^{-i}}{x_i} = \frac{1}{l_{ii}}\boldsymbol{\imath}'(\mathbf{L} - \mathbf{I})\mathbf{e}_i = \frac{(\mathbf{b}' - \boldsymbol{\imath}')\mathbf{e}_i}{l_{ii}} = \frac{b_i - 1}{l_{ii}},$$

which is the first outcome in Result 1.

In order to obtain the closed form expression for the forward linkage f_i^h, we again employ the identity (A1) with $\mathbf{X} = \mathbf{I} - \mathbf{B}$, $\mathbf{u} = \mathbf{e}_i$ and $\mathbf{z}' = \mathbf{e}_i'\mathbf{B}$, and noting that $\mathbf{B}_r^{-i} = (\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')\mathbf{B}$ we obtain

$$
\begin{aligned}
(\mathbf{x}_r^{-i})' &= \mathbf{v}'(\mathbf{I} - \mathbf{B}_r^{-i})^{-1} = \mathbf{v}'(\mathbf{I} - \mathbf{B} + \mathbf{e}_i\mathbf{e}_i'\mathbf{B})^{-1} = \mathbf{v}'\left(\mathbf{G} - \frac{1}{1 + \mathbf{e}_i'\mathbf{B}\mathbf{G}\mathbf{e}_i}\mathbf{G}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{G}\right) \\
&= \mathbf{x}' - \frac{1}{1 + \mathbf{e}_i'(\mathbf{G} - \mathbf{I})\mathbf{e}_i}\mathbf{x}'\mathbf{e}_i\mathbf{e}_i'(\mathbf{G} - \mathbf{I}) = \mathbf{x}' - \frac{x_i}{l_{ii}}\mathbf{e}_i'(\mathbf{G} - \mathbf{I}),
\end{aligned}
$$

using the identity $\mathbf{B}\mathbf{G} = \mathbf{B} + \mathbf{B}^2 + \cdots = \mathbf{G} - \mathbf{I}$, and the fact that the diagonal elements of the Leontief and Ghosh inverses are equal as follows from

$$
\mathbf{L} = (\mathbf{I} - \mathbf{Z}\hat{\mathbf{x}}^{-1})^{-1} = (\mathbf{I} - \hat{\mathbf{x}}\mathbf{B}\hat{\mathbf{x}}^{-1})^{-1} = (\hat{\mathbf{x}}(\mathbf{I} - \mathbf{B})\hat{\mathbf{x}}^{-1})^{-1} = \hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1}, \tag{A2}
$$

that is, $l_{ii} = (x_i g_{ii})/x_i = g_{ii}$ for all i. Thus, the hypothetical extraction forward linkage of sector i is

$$
f_i^h = \frac{\mathbf{x}'\boldsymbol{\imath} - (\mathbf{x}_r^{-i})'\boldsymbol{\imath}}{x_i} = \frac{1}{l_{ii}}\mathbf{e}_i'(\mathbf{G} - \mathbf{I})\boldsymbol{\imath} = \frac{\mathbf{e}_i'(\mathbf{f} - \boldsymbol{\imath})}{l_{ii}} = \frac{f_i - 1}{l_{ii}},
$$

which proves the second outcome in Result 1. □

Derivation of (13). In the first proof of derivation of (8), replace the summation vector $\boldsymbol{\imath}$ by the direct factor coefficients vector $\boldsymbol{\pi}$ and normalize with respect to $\pi_i x_i$. Hence,

$$
b_i^{\pi,h} = \frac{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}_c^{-i}}{\pi_i x_i} = \frac{1}{\pi_i l_{ii}}\boldsymbol{\pi}'(\mathbf{L} - \mathbf{I})\mathbf{e}_i = \frac{(\mathbf{b}_\pi' - \boldsymbol{\pi}')\mathbf{e}_i}{\pi_i l_{ii}} = \frac{b_i^\pi - \pi_i}{\pi_i l_{ii}}.
$$

Similarly, we obtain

$$
f_i^{\pi,h} = \frac{\mathbf{x}'\boldsymbol{\pi} - (\mathbf{x}_r^{-i})'\boldsymbol{\pi}}{\pi_i x_i} = \frac{1}{\pi_i l_{ii}}\mathbf{e}_i'(\mathbf{G} - \mathbf{I})\boldsymbol{\pi} = \frac{\mathbf{e}_i'(\mathbf{f}_\pi - \boldsymbol{\pi})}{\pi_i l_{ii}} = \frac{f_i^\pi - \pi_i}{\pi_i l_{ii}},
$$

which proves Result 3. □

Derivation of (16)-(17). From (11) and (13) it follows that $1/l_{ii} = \omega_i^\pi - b_i^{\pi,h}$. Plugging the last in the formula of $b_i^{\pi,h}$ and solving for $b_i^{\pi,h}$ after simple mathematical transformations gives $b_i^{\pi,h} = \omega_i^\pi(b_i^\pi - \pi_i)/b_i^\pi$, which is the first expression for $b_i^{\pi,h}$ in Result 3. From the factor net backward linkage (14) we have $b_i^\pi = b_i^{\pi,n}\pi_i/s_i$, where $s_i = y_i/x_i$. Substituting this in the last expression gives the second formulation for $b_i^{\pi,h}$. For the forward linkage, we have

$$
f_i^{\pi,h} = (f_i^\pi - \pi_i)\frac{1}{\pi_i l_{ii}} = (f_i^\pi - \pi_i)\frac{\omega_i^\pi - b_i^{\pi,h}}{\pi_i} = (f_i^\pi - \pi_i)\left(\frac{\omega_i^\pi}{\pi_i} - \frac{\omega_i^\pi(b_i^\pi - \pi_i)}{b_i^\pi \pi_i}\right) = \frac{\omega_i^\pi(f_i^\pi - \pi_i)}{b_i^\pi}.
$$

Finally, using (14) gives the second expression for $f_i^{\pi,h}$ in Result 3.

The final expressions in Result 3 for $b_i^{\pi,h}$ and $f_i^{\pi,h}$ can be easily found by using the fact that $b_i^{\pi,n} = \frac{b_i^\pi s_i}{f_i^\pi \tilde{s}_i} f_i^{\pi,n}$ (which, in fact, gives another relation among the given linkages). □